AN EXAMPLE IN THE THEORY OF HYPERCONTRACTIVE SEMIGROUPS

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Abstract. Let $L = x(d^2/dx^2) + (1-x)(d/dx)$ on $C_c((0, \infty))$ be the Laguerre operator. It is shown that for $t > 0$, and $1 < p < q < \infty$, $e^{it} : L^p(e^{-x} dx) \rightarrow L^q(e^{-x} dx)$ has norm 1 if and only if $e^{-t} \leq (p-1)/(q-1)$ and the corresponding logarithmic Sobolev constant is not equal to $2/\lambda$, where $\lambda$ is the smallest nonzero eigenvalue of $L$.

Let $(E, \mathcal{F}, m)$ be a probability space and $(P_t : t > 0)$ a conservative Markov semigroup on $B(E)$ for which $m$ is a reversible measure (i.e. for each $t > 0$, $P_t$ is symmetric on $L^2(m)$). Then, as an easy application of Jensen's inequality, \[ \|P_t\|_{L^p(m) \rightarrow L^p(m)} \leq 1 \] for all $t > 0$ and $p \in [1, \infty]$. In particular, each $P_t$ admits a unique extension $\overline{P}_t$ as a bounded operator on $L^2(m)$ and $(\overline{P}_t : t > 0)$ is a semigroup of selfadjoint contractions. A well-studied example of this situation is the Ornstein-Uhlenbeck semigroup $(\Gamma_t^{(d)} : t > 0)$ on $B(R^d)$: $E = R^d$, $m(dx) = g(d)(l, x) dx$, and $P_t = \Gamma_t^{(d)}$ is given by

\[ \Gamma_t^{(d)} f(x) = \int g^{(d)}(1 - e^{-2t}, y - e^{-t}x) f(y) dy \]

where $g^{(d)}(\tau, \xi) = (2\pi \tau)^{-d/2} \exp(-|\xi|^2/2\tau)$, $(\tau, \xi) \in (0, \infty) \times R^d$. In connection with his work on constructive field theory, E. Nelson [2] discovered that $(\Gamma_t^{(d)} : t > 0)$ enjoys a hypercontractivity property. Namely, he showed that for given $1 < p < q < \infty$, $\|\Gamma_t^{(d)}\|_{L^p(\gamma^{(d)}) \rightarrow L^q(\gamma^{(d)})} \leq 1$ if and only if $e^{-2t} \leq (p-1)/(q-1)$. In addition, he noted that if $e^{-2t} > (p-1)/(q-1)$, then $\|\Gamma_t^{(d)}\|_{L^p(\gamma) \rightarrow L^q(\gamma)} = \infty$. Since Nelson's initial discovery, many other examples of hypercontractive semigroups have been found (cf. F. Weissler [7, 8], F. Weissler and C. Mueller [9], and O. Rothaus [3–5]). In most cases the difficult part of the analysis lies in the attempt to obtain the optimal result (i.e. the smallest $T(p, q) > 0$ such that $\|P_t\|_{L^p(m) \rightarrow L^q(m)} \leq 1$ for all $t \geq T(p, q)$). The work of L. Gross [1] shows that this question is closely related to that of finding the smallest $\alpha > 0$ for which the logarithmic Sobolev inequality

\[ \int |f|^2 \log |f|^2 dm \leq \alpha \mathcal{E}(f, f) + \|f\|_{L^2(m)}^2 \log \|f\|_{L^2(m)}^2 \]

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holds, where \( \mathcal{E} \) denotes the Dirichlet form associated with \( \{ \overline{P}_t; t > 0 \} \) (i.e.,
\[
\mathcal{E}(f, f) = \sup_{t > 0} \frac{1}{t} \left( f - \overline{P}_tf, f \right)_{L^2(m)} = \lim_{t \to 0} \frac{1}{t} \left( f - \overline{P}_tf, f \right)_{L^2(m)}
\]
and \( \text{Dom}(\mathcal{E}) = \{ f \in L^2(m): \mathcal{E}(f, f) < \infty \} \). Indeed, under mild conditions, Gross's analysis shows that (1) for a given \( \alpha > 0 \) is equivalent to
\[
\|P_t\|_{L^p(m) \to L^q(m)} \leq 1, \quad e^{-\alpha t} \leq \frac{p-1}{q-1}.
\]
(cf. D. Stroock [6, §9], for additional information). Further, Rothaus [3] has shown that the logarithmic Sobolev constant (i.e., the smallest \( \alpha \) for which (1) holds) must be at least \( 2/\lambda \), where
\[
\lambda = \inf \left\{ \mathcal{E}(f, f): \|f\|_{L^2(m)} = 1 \text{ and } \int f \, dm = 0 \right\}
\]
is the gap between 0 and the rest of the spectrum of the generator \( \{ \overline{P}_t; t > 0 \} \). For the most part, the technique adopted for proving optimality has been to prove that (1) holds with \( \alpha = 2/\lambda \) (cf. [9]).

The main purpose of this note is to provide a simple example for which the hypercontractivity constant is not \( 2/\lambda \). To this end, take: \( \mathcal{E} = [0, \infty), m(dp) = e^{\rho^2} \, dp \), and for locally bounded measurable \( f: [0, \infty) \to \mathbb{R} \) having subexponential growth at \( \infty \), define \( P_t \) so that
\[
P_t f((x^2)/2) = \left[ \Gamma_{\gamma/2} f \right](\rho \omega), \quad t > 0 \text{ and } \rho \in [0, \infty),
\]
where \( f(x) = f(|x|^2/2), x \in \mathbb{R}^2, \) and \( \omega = (\frac{1}{\sqrt{\pi}}) \in \mathbb{R}^2 \). Then the following facts about \( \{ P_t; t > 0 \} \) are easy to check:

(i) \( \{ P_t|B(E); t > 0 \} \) is a conservative Markov semigroup,
(ii) for each \( t > 0, P_t \) is symmetric on \( L^2(m) \).

(6) Lemma. Let \( 1 < p < q < \infty \) and \( t > 0 \) be given. If \( e^{-t} \leq (p-1)/(q-1) \), then \( \|P_t\|_{L^p(m) \to L^q(m)} \leq 1 \). If \( e^{-t} > (p-1)/(q-1) \), then \( \|P_t\|_{L^p(m) \to L^q(m)} = \infty \).

Proof. Note that for any \( r \in [1, \infty) \) and any measurable \( g: [0, \infty) \to \mathbb{R} \), \( \|g\|_{L^r(m)} = \|\tilde{g}\|_{L^r(\gamma^{(2)})} \). Also, observe that for any locally bounded \( f: [0, \infty) \to \mathbb{R} \) having subexponential growth at \( \infty \), \( \Gamma_{\gamma/2}^{(2)} \tilde{f} = \overline{P}_t \tilde{f}, t > 0 \) Thus, \( \|P_t\|_{L^p(m) \to L^q(m)} \leq 1 \) is equivalent to \( \|\Gamma_{\gamma/2}^{(2)} \tilde{f}\|_{L^q(\gamma^{(2)})} \leq \|\tilde{f}\|_{L^q(\gamma^{(2)})} \) for all locally bounded measurable \( f: [0, \infty) \to \mathbb{R} \), which have subexponential growth at \( \infty \). In particular, by Nelson's inequality, \( \|P_t\|_{L^p(m) \to L^q(m)} \leq 1 \) if \( e^{-t} \leq (p-1)/(q-1) \). To prove that \( \|P_t\|_{L^p(m) \to L^q(m)} = \infty \) if \( e^{-t} > (p-1)/(q-1) \), consider the functions \( f_\lambda(\rho) = \exp(21/2\lambda \rho^{1/2} - \lambda^2/2) \) for \( \lambda > 0 \). In view of the preceding considerations, we need only check that
\[
\lim_{\lambda \to \infty} \left\| \Gamma_{\gamma/2}^{(2)} \tilde{f}_\lambda \right\|_{L^q(\gamma^{(2)})}/\|\tilde{f}_\lambda\|_{L^p(\gamma^{(2)})} = \infty
\]
when \( (p - 1)/(q - 1) > e^{-t} \). By straightforward computation, one can easily see that
\[
\left[ \frac{(\pi/2)^{1/2} r \lambda}{\lambda} \right]^{1/r} \exp(\lambda^2 (r - 1)/2) 
\leq \|f_{\lambda}\|_{L^q(v^{(2)})} \leq \left( 1 + (2\pi)^{1/2} r \lambda \right)^{1/r} \exp(\lambda^2 (r - 1)/2)
\]
for any \( \lambda > 0 \) and \( r \in (1, \infty) \). At the same time,
\[
\left[ \Gamma_t^{(2)} f_{\lambda} \right](x) \geq \sup_{\theta \in S^1} \left[ \Gamma_t^{(2)} g_{\lambda \theta} \right](x) = \sup_{\theta \in S^1} g_{\lambda e^{-t/2} \theta}(x) = f_{\lambda e^{-t/2}}(x),
\]
where \( g_{\eta}(x) = \exp(\eta \cdot x - |\eta|^2/2) \) for \( \eta \in \mathbb{R}^2 \) and we have used the fact that \( \Gamma_s^{(2)} g_{\eta} = g_{e^{-s} \eta} \) for all \( s > 0 \) and \( \eta \in \mathbb{R}^2 \). After combining these, one easily arrives at the desired conclusion. Q.E.D.

To complete our analysis, we must compute the \( \lambda \) associated with \( \{P_t: t > 0\} \). To this end, let \( \{Y_n: n > 0\} \) be the normalized Laguerre polynomials (i.e. the normalized orthogonal polynomials on \([0, \infty)\) with respect to \( m \)) and define \( \gamma = A - \langle v \rangle \) on \( C^\infty(\mathbb{R}^2) \). Then, as is well known,
\[
P^2 = -2nY^n, \quad n > 0.
\]
Since \( \Gamma_t^{(2)} f = f_0 \Gamma_t^{(2)} Hf ds \), \( t > 0 \), for all polynomials \( f: R^2 \to R^1 \), we conclude that
\[
\Gamma_t^{(2)} \tilde{Y}_n = e^{-nt} \tilde{Y}_n
\]
and therefore that
\[
P_t Y_n = e^{-nt} Y_n
\]
for all \( t > 0 \) and \( n > 0 \). As an immediate consequence, we now have that
\[
\bar{P}_t f = \sum_{n=0}^{\infty} e^{-nt}(f, Y_n)_{L_2(m)} Y_n, \quad t > 0 \text{ and } f \in L^2(m).
\]
In particular, the Dirichlet form \( \mathcal{E} \) for \( \{\bar{P}_t: t > 0\} \) is given by
\[
\mathcal{E}(f, f) = \sum_{n=1}^{\infty} n(f, Y_n)_{L_2(m)}^2, \quad f \in L^2(m),
\]
and so the corresponding gap \( \lambda \) is 1.

By combining Gross's analysis, Lemma (6) and the preceding, we now have the following result.

(7) THEOREM. Let \( m(d\rho) = e^{-\rho} d\rho \) on \([0, \infty)\) and define \( P_t, t > 0, \text{ by } (4). \) Then \( \{P_t: t > 0\} \) is a conservative Markov semigroup which is symmetric in \( L^2(m) \). Let \( \{P_t: t > 0\} \) be the semigroup of \( L^2(m) \)-selfadjoint contractions determined by \( \{P_t: t > 0\} \) and denote by \( \mathcal{E} \) the associated Dirichlet form. Then
\[
1 = \inf \{ \mathcal{E}(f, f): f \in L^2(m), \|f\|_{L_2(m)} = 1 \text{ and } \int f dm = 0 \},
\]
On the other hand, the logarithmic Sobolev constant for \( \mathcal{E} \) (i.e. the smallest \( \alpha \) for which (2) holds) is 4.
Remark. The semigroup \( \{ P_t; \ t > 0 \} \) in Theorem (7) can be described directly in terms of the Laguerre operator

\[
L = \rho \frac{d^2}{d\rho^2} + (1 - \rho) \frac{d}{d\rho} \quad \text{on} \quad C^\infty_c((0, \infty)).
\]

Indeed, \( \{ P_t; \ t > 0 \} \) is the unique conservative Markov semigroup on \( B((0, \infty)) \) such that

\[
P_tf - f = \int_0^t P_s Lf \, ds, \quad t \geq 0,
\]

for all \( f \in C^\infty_c((0, \infty)) \). Thus there are several reasons for calling \( \{ P_t; \ t > 0 \} \) the Laguerre semigroup. In this connection it is natural to suspect that the reason why, in this example, the logarithmic Sobolev constant \( \alpha_0 \) and the spectral gap \( \lambda \) do not satisfy \( \alpha_0 = 2/\lambda \) may have something to do with the way in which \( L \) degenerates at 0.

References