AN EXAMPLE IN THE THEORY
OF HYPERCONTRACTIVE SEMIGROUPS

ANDRZEJ KORZENIOWSKI AND DANIEL W. STROOCK

Abstract. Let \( L = x \frac{d^2}{dx^2} + (1-x) \frac{d}{dx} \) on \( C_c((0, \infty)) \) be the Laguerre operator. It is shown that for \( t > 0 \), and \( 1 < p < q < \infty \), \( e^{it} : L^p(e^{-x}dx) \to L^q(e^{-x}dx) \) has norm 1 if and only if \( e^{-it} \leq (p-1)/(q-1) \) and the corresponding logarithmic Sobolev constant is not equal to \( 2/\lambda \), where \( \lambda \) is the smallest nonzero eigenvalue of \( L \).

Let \((E, \mathcal{F}, m)\) be a probability space and \( \{P_t : t > 0\} \) a conservative Markov semigroup on \( B(E) \) for which \( m \) is a reversible measure (i.e. for each \( t > 0 \), \( P_t \) is symmetric on \( L^2(m) \)). Then, as an easy application of Jensen's inequality, 
\[
\|P_t\|_{L^p(m) \to L^p(m)} \leq 1
\]
for all \( t > 0 \) and \( p \in [1, \infty] \). In particular, each \( P_t \) admits a unique extension \( \overline{P_t} \) as a bounded operator on \( L^2(m) \) and \( \{\overline{P_t} : t > 0\} \) is a semigroup of selfadjoint contractions. A well-studied example of this situation is the Ornstein-Uhlenbeck semigroup \( \{\Gamma_t^{(d)} : t > 0\} \) on \( B(R^d) : E = R^d, m(dx) = \gamma(dx) = g(d)(l, x) dx \), and \( P_t = \Gamma_t^{(d)} \) is given by
\[
\Gamma_t^{(d)} f(x) = \int g^{(d)}(1 - e^{-2t}, y - e^{-t}x) f(y) dy
\]
where \( g^{(d)}(\tau, \xi) = (2\pi\tau)^{-d/2} \exp(-|\xi|^2/2\tau) \), \( (\tau, \xi) \in (0, \infty) \times R^d \). In connection with his work on constructive field theory, E. Nelson [2] discovered that \( \{\Gamma_t^{(d)} : t > 0\} \) enjoys a hypercontractivity property. Namely, he showed that for given \( 1 < p < q < \infty \), \( \|\Gamma_t^{(d)}\|_{L^p(\gamma^{(d)}) \to L^q(\gamma^{(d)})} \leq 1 \) if and only if \( e^{-2t} \leq (p-1)/(q-1) \). In addition, he noted that if \( e^{-2t} > (p-1)/(q-1) \), then \( \|\Gamma_t^{(d)}\|_{L^p(\gamma) \to L^q(\gamma)} = \infty \).

Since Nelson's initial discovery, many other examples of hypercontractive semigroups have been found (cf. F. Weissler [7, 8], F. Weissler and C. Mueller [9], and O. Rothaus [3-5]). In most cases the difficult part of the analysis lies in the attempt to obtain the optimal result (i.e. the smallest \( T(p, q) > 0 \) such that \( \|P_t\|_{L^p(m) \to L^q(m)} \leq 1 \) for all \( t \geq T(p, q) \)). The work of L. Gross [1] shows that this question is closely related to that of finding the smallest \( \alpha > 0 \) for which the logarithmic Sobolev inequality
\[
\int |f|^2 \log |f|^2 dm \leq \alpha \mathcal{E}(f, f) + \|f\|_{L^2(m)}^2 \log \|f\|_{L^2(m)}^2
\]
is satisfied.

Received by the editors March 27, 1984 and, in revised form, June 6, 1984.
1980 Mathematics Subject Classification. Primary 47D05; Secondary 46E30.
Key words and phrases. Laguerre semigroup, logarithmic Sobolev inequality, Ornstein-Uhlenbeck semigroup, hypercontractivity.

1The work of this author was supported in part by N.S.F. Grant MCS 8310542.

©1985 American Mathematical Society
0002-9939/85 $1.00 + $.25 per page
holds, where $\mathcal{E}$ denotes the Dirichlet form associated with $\{P_t: t > 0\}$ (i.e.,
\[ \mathcal{E}(f, f) = \sup_{t > 0} \frac{1}{t} (f - P_t f, f)_{L^2(m)} = \lim_{t \to 0} \frac{1}{t} (f - P_t f, f)_{L^2(m)} \]
and $\text{Dom}(\mathcal{E}) = \{f \in L^2(m): \mathcal{E}(f, f) < \infty\}$). Indeed, under mild conditions, Gross's analysis shows that (1) for a given $\alpha > 0$ is equivalent to
\[ (1/II P_t II - 1 \alpha^{-1} \leq 1, \quad P_t \text{ is skew-symmetric on } L^\alpha(m) \] (cf. D. Stroock [6, §9], for additional information). Further, Rothaus [3] has shown that the logarithmic Sobolev constant (i.e., the smallest $\alpha$ for which (1) holds) must be at least $2/\lambda$, where
\[ (3) \quad \lambda = \inf \{ \mathcal{E}(f, f): \|f\|_{L^2(m)} = 1 \text{ and } \int f \, dm = 0 \} \]
is the gap between 0 and the rest of the spectrum of the generator $\{P_t: t > 0\}$. For the most part, the technique adopted for proving optimality has been to prove that (1) holds with $\alpha = 2/\lambda$ (cf. [9]).

The main purpose of this note is to provide a simple example for which the hypercontractivity constant is not $2/\lambda$. To this end, take: $E = [0, \infty)$, $m(d\rho) = e^{\rho} \, d\rho$, and for locally bounded measurable $f: [0, \infty) \to \mathbb{R}^1$ having subexponential growth at $\infty$, define $P_t f$ so that
\[ (4) \quad P_t f(\rho^2/2) = \left[ \frac{r}{\rho} \right] \left( \frac{\rho}{r} \right) \frac{\rho^2}{2}, \quad t > 0 \text{ and } \rho \in [0, \infty), \]
where $\tilde{f}(x) = f(|x|^2/2), x \in \mathbb{R}^2, \text{ and } \omega = \left( \frac{1}{r} \right) \in \mathbb{R}^2$. Then the following facts about $\{P_t: t > 0\}$ are easy to check:
\[ (i) \quad \{P_t|_{B(E)}: t > 0\} \text{ is a conservative Markov semigroup,} \]
\[ (ii) \quad \text{for each } t > 0, P_t \text{ is symmetric on } L^2(m). \]

\begin{lemma}
Let $1 < p < q < \infty$ and $t > 0$ be given. If $e^{-t} \leq (p - 1)/(q - 1)$, then $\|P_t\|_{L^p(m) \to L^q(m)} \leq 1$. If $e^{-t} > (p - 1)/(q - 1)$, then $\|P_t\|_{L^p(m) \to L^q(m)} = \infty$.
\end{lemma}

\begin{proof}
Note that for any $r \in [1, \infty)$ and any measurable $g: [0, \infty) \to \mathbb{R}^1$, $\|g\|_{L^r(m)} = \|\tilde{g}\|_{L^r(\gamma^{(2)})}$. Also, observe that for any locally bounded $f: [0, \infty) \to \mathbb{R}^1$ having subexponential growth at $\infty$, $\Gamma_{t, 2}^{(2)} \tilde{f} = P_t \tilde{f} > 0$. Thus, $\|P_t\|_{L^p(m) \to L^q(m)} \leq 1$ is equivalent to $\|\Gamma_{t, 2}^{(2)} \tilde{f}\|_{L^q(\gamma^{(2)})} \leq \|\tilde{f}\|_{L^q(\gamma^{(2)})}$ for all locally bounded measurable $f: [0, \infty) \to \mathbb{R}^1$ which have subexponential growth at $\infty$. In particular, by Nelson's inequality, $\|P_t\|_{L^p(m) \to L^q(m)} \leq 1$ if $e^{-t} \leq (p - 1)/(q - 1)$. To prove that $\|P_t\|_{L^p(m) \to L^q(m)} = \infty$ if $e^{-t} > (p - 1)/(q - 1)$, consider the functions $f_\lambda(\rho) = \exp(21/2 \lambda^2 \rho^2 - \lambda^2/2)$ for $\lambda > 0$. In view of the preceding considerations, we need only check that
\[
\lim_{\lambda \to \infty} \frac{\Gamma_{t, 2}^{(2)} \tilde{f}_\lambda}{\|\tilde{f}_\lambda\|_{L^q(\gamma^{(2)})}} = \infty
\]
when \((p - 1)/(q - 1) > e^{-t}\). By straightforward computation, one can easily see that
\[
\left(\frac{\pi}{2}\right)^{1/2} r \lambda \exp\left(\lambda^2 (r - 1)/2\right) \leq \|f_\lambda\|_{L^q(m)} \leq \left(1 + \left(2\pi\right)^{1/2} r \lambda\right)^{1/r} \exp\left(\lambda^2 (r - 1)/2\right)
\]
for any \(\lambda > 0\) and \(r \in (1, \infty)\). At the same time,
\[
\left[\Gamma_{t/2} f_\lambda\right](x) \geq \sup_{\theta \in S^1} \left[\Gamma_{t/2} g_{e^{i\theta}}\right](x) = \sup_{\theta \in S^1} g_{e^{i\theta}}(x) = f_{e^{i\theta}}(x),
\]
where \(g_\eta(x) = \exp(\eta \cdot x - |\eta|^2/2)\) for \(\eta \in \mathbb{R}^2\) and we have used the fact that \(\Gamma_{s}(g_\eta) = g_{e^{-s}\eta}\) for all \(s > 0\) and \(\eta \in \mathbb{R}^2\). After combining these, one easily arrives at the desired conclusion. Q.E.D.

To complete our analysis, we must compute the \(\lambda\) associated with \(\{P_t: t > 0\}\). To this end, let \(\{Y_n: n > 0\}\) be the normalized Laguerre polynomials (i.e. the normalized orthogonal polynomials on \([0, \infty)\) with respect to \(m\)) and define \(Y_n = \mathbb{R}^2\) on \(C^\infty(\mathbb{R}^2)\). Then, as is well known,
\[
\begin{align*}
\rho \frac{d^2 Y_n}{d \rho^2}(\rho) + (1 - \rho) \frac{d Y_n}{d \rho}(\rho) &= -n Y_n(\rho), \quad n \geq 0 \text{ and } \rho \in [0, \infty).
\end{align*}
\]
From this, it is an easy matter to check that
\[
H \tilde{Y}_n = -2n \tilde{Y}_n, \quad n \geq 0.
\]
Since \(\Gamma_{t/2} f = \int_0^t \Gamma_s^{(2)} H f \, ds, \quad t > 0,\) for all polynomials \(f: \mathbb{R}^2 \to \mathbb{R}\), we conclude that
\[
\Gamma_{t/2} \tilde{Y}_n = e^{-nt} \tilde{Y}_n
\]
and therefore that
\[
P_t Y_n = e^{-nt} Y_n
\]
for all \(t > 0\) and \(n \geq 0\). As an immediate consequence, we now have that
\[
P_t f = \sum_{n=0}^\infty e^{-nt}(f, Y_n)_{L^2(m)} Y_n, \quad t > 0 \text{ and } f \in L^2(m).
\]
In particular, the Dirichlet form \(\mathcal{E}\) for \(\{P_t: t > 0\}\) is given by
\[
\mathcal{E}(f, f) = \sum_{n=1}^\infty n(f, Y_n)_{L^2(m)}^2, \quad f \in L^2(m),
\]
and so the corresponding gap \(\lambda\) is 1.

By combining Gross's analysis, Lemma (6) and the preceding, we now have the following result.

(7) THEOREM. Let \(m(d\rho) = e^{-\rho} d\rho\) on \([0, \infty)\) and define \(P_t, t > 0,\) by (4). Then \(\{P_t: t > 0\}\) is a conservative Markov semigroup which is symmetric in \(L^2(m)\). Let \(\{P_t: t > 0\}\) be the semigroup of \(L^2(m)\)-selfadjoint contractions determined by \(\{P_t: t > 0\}\) and denote by \(\mathcal{E}\) the associated Dirichlet form. Then
\[
1 = \inf \left\{ \mathcal{E}(f, f) : f \in L^2(m), \|f\|_{L^2(m)} = 1 \text{ and } \int f \, dm = 0 \right\},
\]
On the other hand, the logarithmic Sobolev constant for \(\mathcal{E}\) (i.e. the smallest \(\alpha\) for which (2) holds) is 4.
Remark. The semigroup \( \{ P_t : t > 0 \} \) in Theorem (7) can be described directly in terms of the Laguerre operator

\[
L = \rho \frac{d^2}{d\rho^2} + (1 - \rho) \frac{d}{d\rho} \quad \text{on} \; C_\infty^\infty((0, \infty)).
\]

Indeed, \( \{ P_t : t > 0 \} \) is the unique conservative Markov semigroup on \( C_\infty^\infty((0, \infty)) \) such that

\[
P_t f - f = \int_0^t P_s L f \, ds, \quad t \geq 0,
\]

for all \( f \in C_\infty^\infty((0, \infty)) \). Thus there are several reasons for calling \( \{ P_t : t > 0 \} \) the Laguerre semigroup. In this connection it is natural to suspect that the reason why, in this example, the logarithmic Sobolev constant \( \alpha_0 \) and the spectral gap \( \lambda \) do not satisfy \( \alpha_0 = 2/\lambda \) may have something to do with the way in which \( L \) degenerates at 0.

References


Department of Mathematics, University of Texas, Arlington, Texas 76019

Department of Mathematics, University of Colorado, Boulder, Colorado 80209

Current address (D. W. Stroock): Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139