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GENERIC FRÉCHET DIFFERENTIABILITY OF CONVEX OPERATORS

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ABSTRACT. We consider order-bounded convex operators $F\colon E \to X$ from a reflexive Banach space E into a Banach lattice X. In both cases (i) X and X^* have weak compact intervals, and (ii) X has norm compact intervals, we obtain that F is Fréchet differentiable at the points of some dense G_δ subset of E.

0. Introduction. Let E be a Banach space, X a Banach lattice, and $L(E, X)$ the space of all continuous linear operators from E to X. Let $K(E, X)$ denote the space of compact linear operators. We recall that $F\colon E \to X$ is called a convex operator if

$$F(\lambda e_1 + (1-\lambda)e_2) \leqslant \lambda Fe_1 + (1-\lambda)Fe_2$$

whenever $e_1, e_2 \in E$ and $0 \leqslant \lambda \leqslant 1$. F is called order-bounded if for each $e_0 \in E$ there exists a neighbourhood V of e_0 such that $F(V)$ is an order-bounded subset of X.

In this paper we study the problem of generic Fréchet differentiability of convex operators defined in E; i.e., we give conditions under which every order-bounded convex operator F is Fréchet differentiable at the points of some dense G_δ subset of E. In the case when F is a real-valued convex function (i.e., $X = R$), this problem has been thoroughly studied. Banach spaces E for which every continuous convex real-valued function $F\colon E \to R$ is Fréchet differentiable at the points of a dense G_δ subset of E are called Asplund spaces. These spaces have been chracterized in many different ways (see Namioka and Phelps [**14**], Phelps [**15**], Stegal [**16**], Kenderov [**10**]).

In another particular case, namely when $E = R^n$, the author found [**13**] the necessary and sufficient condition for generic Fréchet differentiability of convex operators $F\colon E \to X$. This condition is X has weak compact intervals (see also Borwein [**3**]).

There are some results in the cases when both E and X are infinite dimensional. Two of these results can be found in [**13**]. If E is an arbitrary Banach space, X is a dual lattice such that $L(E, X)$ has the Radon-Nikodym property (for the defintion see Diestel and Uhl [**6**, p. 61]). Then every continuous convex operator $F\colon E \to X$ is generic Fréchet differentiable. Sufficient conditions for $L(E, X)$ to have the Radon-Nikodým property are the following: the space E^* and the dual lattice X have the Radon-Nikodým property, and $L(E, X) = K(E, X)$ (Andrews [**1**]). The

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assumptions in the second result, given in the paper mentioned above, are X has norm compact intervals, F is order-bounded, and $L(E, X) = (K(E, X))^{**}$ (the bidual space). Different types of conditions, including a quite restrictive supposition for the positive cone of X, were obtained by Borwein [**2**, Theorem 5.2], to prove that every continuous convex operator from an Asplund space E to X is generic Fréchet differentiable.

It should be noted here that the case of arbitrary E, X, and F is very complicated. There is a continuous convex operator $P\colon l_2 \to l_2$, where l_2 is the usual Hilbert space, which is nowhere Fréchet differentiable in l_2 (see [**12**, Example 2]). This shows that we must impose some kind of restrictions on the operator F in order to have positive results. Order boundedness, first used by Valadier [**17**], then by the author [**13**], is the most natural restriction and it will also be used here.

Furthermore, we shall employ Kenderov's method of multivalued monotone mappings (see [**9–11**]), and for this purpose we introduce some more definitions.

We say that the multivalued mapping $T\colon E \to L(E, X)$ is a generalized monotone mapping (briefly g.m.m.) if $(A_1 - A_2)(e_1 - e_2) \geqslant 0$ for every $A_i \in Te_i$, $e_i \in E$, $i = 1, 2$. The subdifferential of a convex operator $F\colon E \to X$, i.e., the multivalued mapping

$$\partial_F\colon e_0 \mapsto \{A \in L(E, X)\colon A(e - e_0) \leqslant Fe - Fe_0 \text{ for every } e \in E\}$$

is a g.m.m. The g.m.m. is said to be maximal if its graph is not properly contained in the graph of any other g.m.m. By Zorn's lemma the graph of every g.m.m. is contained in the graph of some maximal g.m.m. In what follows we suppose that $Te \neq \varnothing$ for all $e \in E$.

The multivalued mapping $T\colon E \to L$ is said to be upper semicontinuous at the point $e_0 \in E$ if for every open set $U \supset Te_0$ there exists a neighbourhood W of e_0 such that $Te \subset U$ for all $e \in W$.

1. Preliminary results. It is known [**12**, Theorem 4] that the continuous convex operator $F\colon E \to X$ is Fréchet differentiable at $e \in E$ if and only if the g.m.m. ∂_F is single-valued and norm-to-norm upper semicontinuous at this point. This fact, together with a corollary of a topological result of Christensen [**5**, Theorem 2], gives us the desired result.

THEOREM 1.1 (CHRISTENSEN). *Let E be a Banach space, Z a normed space, and T: $E \to Z$ a norm-to-weak upper semicontinuous multivalued mapping with nonempty and weak compact images. Then there exists a dense G_δ subset of E at each point e_0 of which the following condition is fulfilled*:

(∗) *There exists a point $z_0 \in Te_0$ such that for every $\varepsilon > 0$ there is $\delta > 0$ with $\inf\{\|z - z_0\|\colon z \in Te\} \leqslant \varepsilon$ whenever $\|e - e_0\| < \delta$.*

PROPOSITION 1.2. *Let E be a Banach space, X a normed lattice, and T: $E \to L(E, X)$ a g.m.m. Then T has property (∗) at the point $e_0 \in E$ iff T is single-valued and norm-to-norm upper semicontinuous at e_0.*

PROOF. It is easily seen that if T is single-valued and norm-to-norm upper semicontinuous at e_0, then T has property $(*)$ at e_0. Suppose T has $(*)$ at e_0. Choosing the point $A_0 \in Te_0$ from $(*)$ and some $\varepsilon > 0$, we see that there exists $\delta > 0$ such that $\inf\{\|A - A_0\|: A \in Te\} < \varepsilon/2$ whenever $e \in U = \{e \in E: \|e - e_0\| < \delta\}$. We shall prove that $Te \subset \{A \in L(E, X): \|A - A_0\| < \varepsilon\}$ for any $e \in U$. That is sufficient to conclude that T is single-valued and norm-to-norm upper semicontinuous at e_0. Let $e \in U$, $A \in Te$, $h \in E$, $\|h\| \leqslant 1$. For some $t > 0$ we have $e' = e + th \in U$ and $e'' = e - th \in U$. There are $A' \in Te'$ and $A'' \in Te''$ such that $\|A' - A_0\| < \varepsilon/2$ and $\|A'' - A_0\| < \varepsilon/2$. By the monotonicity of T we get $A'h \geqslant Ah$ and $Ah \geqslant A''h$. Hence,

$$(A'' - A_0)h \leqslant (A - A_0)h \leqslant (A' - A_0)h.$$

Let $x^* \in X^*_+$ (the positive cone of X^*), $\|x^*\| \leqslant 1$. We have

$$\langle (A'' - A_0)h, x^* \rangle \leqslant \langle (A - A_0)h, x^* \rangle \leqslant \langle (A' - A_0)h, x^* \rangle.$$

Consequently, $\langle (A - A_0)h, x^* \rangle \leqslant \varepsilon/2$. Since X^* is a lattice, $|\langle (A - A_0)h, x^* \rangle| \leqslant \varepsilon$ for any $x^* \in X^*$, $\|x^*\| \leqslant 1$. It follows that $\|(A - A_0)h\| \leqslant \varepsilon$, and the proposition is proved.

Now it becomes clear that if the g.m.m. ∂_F satisfies the suppositions of Theorem 1.1, then F is generic Fréchet differentiable. In order to apply this theorem for a g.m.m. $T: E \to L(E, X)$, we must choose some suitable closed linear subspace $Z \subset L(E, X)$ such that $Te \subset Z$ for all $e \in E$. Then we must prove that T satisfies the requirement of the theorem. Before presenting the proposition we need a definition.

We say that the g.m.m. T is locally weak-order bounded if for every $e_0 \in E$ there exists a neighbourhood V of e_0 (with respect to the norm topology) such that the set $T(V) = \bigcup\{Te: e \in V\}$ is weak-order bounded. This means that the sets $\{Ae: A \in T(V)\}$ are order bounded for every $e \in E$.

PROPOSITION 1.3 [12]. *Let E be a Banach space, X a Banach lattice with weak compact intervals, and $T: (E, \|\cdot\|) \to (L(E, X), \sigma(L(E, X), E \otimes X^*))$ a locally weak-order bounded and maximal g.m.m. Then T is upper semicontinuous and compact valued at all points of E.*

REMARK. We recall that the tensor product $E \otimes X^*$ can be considered as a subspace of $L(E, X)^*$. Every point $u \in E \otimes X^*$, $u = \sum_{i=1}^k e_i \otimes x_i^*$, defines a continuous linear functional on $L(E, X)$ in the following way: $\langle A, u \rangle = \sum_{i=1}^k \langle Ae_i, x_i^* \rangle$ for all $A \in L(E, X)$.

PROOF. Since X has weak compact intervals, every weak-order bounded subset of $L(E, X)$ is relatively compact in the topology $\sigma(L(E, X), E \otimes X^*)$. Hence, for every $e_0 \in E$ there is an open set $V \ni e_0$ such that $T(V)$ is relatively compact, because T is locally weak-order bounded. It is not difficult to prove that such a mapping is upper semicontinuous and compact valued, provided it has closed graph (see [13, Proposition 1.5]). That every maximal g.m.m. has a closed graph can be seen from the following lemma.

LEMMA 1.4. *Every maximal g.m.m. has a closed graph with respect to the topology* $\sigma(L(E, X), E \otimes X^*)$.

PROOF. Let $e_0 \in E$, $\{e_\alpha\}$ be a net, $\|e_\alpha - e_0\| \to 0$, $A_\alpha \in Te_\alpha$, and $A_\alpha \to A_0$ in the topology described above. That means $\langle A_\alpha e, x^*\rangle \to \langle A_0 e, x^*\rangle$ for every $e \in E$ and $x^* \in X^*$. We must show that $A_0 \in Te_0$. For each $e \in E$ and $A \in Te$,

$$(A - A_\alpha)(e - e_\alpha) = Ae - A_\alpha e - Ae_\alpha + A_\alpha e_\alpha \geqslant 0$$

by monotonicity. Now $A_\alpha e \to A_0 e$ in the weak topology of X and $\|Ae_\alpha - Ae_0\| \to 0$. If $x^* \in X^*$,

$$\begin{aligned}|\langle A_\alpha e_\alpha - A_0 e_0, x^*\rangle| &\leqslant |\langle A_\alpha e_\alpha - A_\alpha e_0, x^*\rangle| + |\langle A_\alpha e_0 - A_0 e_0, x^*\rangle| \\ &\leqslant \|A_\alpha\|\,\|e_\alpha - e_0\|\,\|x^*\| + |\langle (A_\alpha - A_0)e_0, x^*\rangle| \to 0,\end{aligned}$$

because the set $\{A_\alpha\}_{\alpha > \alpha_0}$ is norm bounded [**12**,Theorem 3] for some α_0. Thus, $(A - A_\alpha)(e - e_\alpha) \to (A - A_0)(e - e_0)$ in the weak topology of X. Since the positive cone of each Banach lattice is weakly closed, $(A - A_0)(e - e_0) \geqslant 0$ for every $e \in E$, $A \in Te$. By the maximality of T we obtain that $A_0 \in Te_0$. The proof is finished.

2. G.m.m. with images in $K(E, X)$. In this section we assume that every g.m.m. has images in the space of compact operators $K(E, X)$.

THEOREM 2.1. *Let E be a reflexive Banach space, X a Banach lattice with weak compact intervals, and T: $E \to K(E, X)$ a locally weak-order bounded g.m.m. Then T is single-valued and norm-to-norm upper semicontinuous at the points of some dense G_δ subset of E.*

PROOF. Proposition 1.3 shows that T is upper semicontinuous and compact valued with respect to the weak topology of $K(E, X)$, because the topologies of $\sigma(K(E, X), E \otimes X^*)$ and $\sigma(K(E, X), K(E, X)^*)$ coincide on the bounded sets (Kalton [**8**]). Theorem 1.1 and Proposition 1.2 can be applied.

COROLLARY 2.2. *If E and X are as in Theorem 2.1 and F: $E \to X$ is an order-bounded convex operator with ∂_F: $E \to K(E, X)$, then F is generic Fréchet differentiable.*

PROOF. This is immediate from Theorem 2.1, because the subdifferential of an order-bounded convex operator is a locally weak-order bounded g.m.m., and single-valuedness and norm-to-norm upper semicontinuity of ∂_F are equivalent to Fréchet differentiability of F.

A convex operator P: $E \to X$ is said to be sublinear if $P(\lambda e) = \lambda Pe$ for all $e \in E$ and $\lambda \in R$, $\lambda \geqslant 0$. Since the subdifferential of every sublinear operator is a locally weak-order bounded g.m.m., provided X has weak compact intervals, we obtain the following:

COROLLARY 2.3. *Suppose E and X are as in Theorem 2.1 and P: $E \to X$ is a sublinear operator with support set (i.e., $\partial_P(0)$) consisting of compact operators. Then P is generic Fréchet differentiable.*

COROLLARY 2.4. *If E is a reflexive space, X is a Banach lattice with norm compact intervals, and F: $E \to X$ is an order-bounded convex operator, then F is generic Fréchet differentiable.*

PROOF. It is not difficult to see that, for every e_0, $e_1 \in E$, there exists a neighbourhood (in the norm topology) W of e_1 such that the set $\{Aw: A \in \partial_F(e_0), w \in W\}$ is order bounded. Hence, $\partial_F(e_0) \subset K(E, X)$ for every $e_0 \in E$. It follows that Corollary 2.2 can be applied.

3. G.m.m. with images in $B(E, X)$. A linear operator A: $E \to X$ is called order bounded if A maps the unit ball of E in an order-bounded subset of X. The space of all order-bounded linear operators will be denoted by $B(E, X)$. It is easy to see that the subdifferential of an order-bounded convex operator has images in $B(E, X)$. That is why we assume T: $E \to B(E, X)$.

Following Heinrich, Nielsen, and Olsen [7], we define a norm in $B(E, X)$:

$$\|A\|_m = \inf\{\|z\|: z \in X \text{ and } |Ae| \leqslant z\|e\| \text{ for every } e \in E, \|e\| = 1\}.$$

It is clear that this m-norm is stronger than the usual operator norm (inherited by $L(E, X)$).

LEMMA 3.1 [7]. *If E is a reflexive space and X and X^* have weak compact intervals, then $(B(E, X), m)^* = E \,\overline{\otimes}_m X^*$, where $E \,\overline{\otimes}_m X^*$ denotes the completion of the tensor product $E \otimes X^*$ under the following crossnorm:*

$$\|u\|_m = \sup\{|\langle A, u\rangle|: A \in B(E, X), \|A\|_m \leqslant 1\}$$

for which $u \in E \otimes X^*$.

THEOREM 3.2. *Let E be a reflexive Banach space, X a Banach lattice such that X and X^* have weak compact intervals, and T: $E \to B(E, X)$ a locally weak-order bounded g.m.m. Then T is single-valued and norm-to-norm upper semicontinuous at the points of some dense G_δ subset of E.*

PROOF. Lemma 3.1 shows that Proposition 1.3 can be applied. Furthermore, the arguments are the same as in the proof of Theorem 2.1.

If X is an order complete Banach lattice, it is interesting to note that the requirement "X and X^* have weak compact intervals" is equivalent to "X is an Asplund space" (Buchvalov, Veksler, and Losanowski [4]).

COROLLARY 3.3. *If E and X are as in Theorem 3.2 and F: $E \to X$ is an order-bounded convex operator, then F is generic Fréchet differentiable.*

PROOF. It is a direct consequence of Theorem 3.2.

4. Examples. Finally, we give some simple examples which outline the field of application of the results. First, we note that the spaces $K(E, X)$ and $B(E, X)$ are generally different. It is easy to see that $B(E, X) \subset K(E, X)$ provided X has norm compact intervals. The linear operator A: $l_p \to l_p$ $(1 \leqslant p < \infty)$, defined by $A(\xi_1, \xi_2, \dots, \xi_k, \dots) = (\xi_1\eta_1, \xi_2\eta_2, \dots, \xi_k\eta_k, \dots)$, where $(\eta_1, \eta_2, \dots, \eta_k, \dots) \in c_0 \setminus l_p$, is compact but not order bounded. An example of a linear operator which belongs to

$B(E, X) \setminus K(E, X)$ is the natural embedding I: $L_\infty[0,1] \to L_p[0,1]$ $(1 \leqslant p < \infty)$. Second, we note that Corollary 3.3 gives us some new pairs of Banach spaces E and Banach lattices X such that every order-bounded convex operator F: $E \to X$ is generic Fréchet differentiable. In particular, for such a pair, E is a reflexive space and $X = L_q$ $(1 < q < \infty)$. In this case Corollary 2.4 does not work because the intervals of X are not compact. Conversely, if E is a reflexive space and $X = l_1 \times l_2$, Corollary 2.4 works (X has norm compact intervals), but Corollary 3.3 does not (X is not Asplund). In both cases no other known results can be applied.

REMARK. Instead of Theorem 1.1 we can use a theorem of Kenderov [**9**, Theorem 2.1]. For g.m.m. both theorems give us the same results.

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