PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 94, Number 1, May 1985

ASYMPTOTIC BEHAVIOR OF ITERATES OF NONEXPANSIVE MAPPINGS IN BANACH SPACES WITH OPIAL'S CONDITION

GIOVANNI EMMANUELE[1]

ABSTRACT. We study the asymptotic behavior of the sequence of the iterates for a nonexpansive mapping, defined on a suitable subset of a Banach space with Opial's condition. Some results are stated also for semigroups of nonexpansive mappings and for mappings of asymptotically nonexpansive type in uniformly convex Banach spaces with Opial's condition.

1. Introduction. Let E be a Banach space. We suppose that, if $\{x_n\} \subseteq E$, $x^0 \in E$ are such that $x_n \xrightarrow{w} x^0$, then

$$(1) \qquad \limsup_n \|x_n - x^0\| < \limsup_n \|x_n - x\|, \qquad x \neq x^0, x \in E;$$

the above condition is known as Opial's condition (see [**12**]).

If X is a boundedly weakly compact subset of E, i.e. a subset of E such that the intersection with each closed ball is weakly compact, we consider a mapping f: $X \to X$ which is nonexpansive.

The purpose of this note is to generalize (§2) a result by Miyadera [**10**] about asymptotic behavior of the sequence $\{f^n(x)\}$. A similar result for uniformly convex Banach spaces with a Frechet differentiable norm has been recently obtained by Kobayashi (see [**9**]); it is known that there are Banach spaces which satisfy Kobayashi's assumptions and not Opial's condition; for example the spaces L^p, $p \neq 2$, $1 < p < \infty$; on the other hand, our Theorem 1 can be applied to spaces which are not uniformly convex (see below).

In §3 we obtain some corollaries of our Theorem 1, whereas in §4 we extend some of our results to the case of semigroups of nonexpansive mappings. In §5 we consider mappings of asymptotically nonexpansive type.

Received by the editors February 3, 1982 and, in revised form, November 11, 1982 and September 1, 1983.

1980 *Mathematics Subject Classification*. Primary 47H10.

Key words and phrases. Opial's condition, nonexpansive mappings, asymptotic behavior of iterates, weak convergence of iterates, uniformly convex in every direction Banach spaces, semigroups of nonexpansive mappings, functions of asymptotically nonexpansive type, metric projection in uniformly convex Banach spaces.

[1] Work performed under the auspices of C.N.R.-G.N.A.F.A.

2. Asymptotic behavior. Under the assumptions of the Introduction we show the following

THEOREM 1. *Let E, X, f be as above. Then the following conditions are equivalent*:

(1) $\{f^n(x)\}$ *converges weakly*;

(ii) $F(f)$, *the fixed point set of* f, *is nonempty and* $\omega_w(x)$, *the set of the weak subsequential limits of* $\{f^n(x)\}$, *is a nonempty subset of* $F(f)$;

(iii) $E(x) \neq \varnothing$ *and* $\omega_w(x) \subseteq E(x)$, *where* $E(x) = \{y\colon y \in X,\ \lim_n \|x_n - y\|$ *exists*$\}$.

PROOF. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) can be obtained easily as in **[10]**. And so we have to show only (iii) $\Rightarrow$ (i). For our purpose we observe that $\{f^n(x)\}$ is bounded since $E(x) \neq \varnothing$; thus $\omega_w(x) \neq \varnothing$. Let y_1, y_2 be two weak subsequential limits for the sequence $\{f^n(x)\}$; there are $d_1, d_2 \geqslant 0$ for which

$$d_1 = \lim_n \|f^n(x) - y_1\|, \qquad d_2 = \lim_n \|f^n(x) - y_2\|.$$

If $\{f^{n(j)}(x)\}$ and $\{f^{n(h)}(x)\}$ are such that $f^{n(j)}(x) \xrightarrow{w} y_1$ and $f^{n(h)}(x) \xrightarrow{w} y_2$, we have that $d_1 \geqslant d_2$ contradicts Opial's condition, since

$$\lim_j \|f^{n(j)}(x) - y_1\| = d_1 \geqslant d_2 = \lim_j \|f^{n(j)}(x) - y_2\|;$$

in a similar way, $d_1 \leqslant d_2$ is false. Thus, $\omega_w(x)$ is a singleton.

The proof is complete.

Our Theorem 1 extends the following result due to Miyadera **[10]**.

COROLLARY 1. *Let E be a smooth, uniformly convex Banach space with a duality mapping which is weakly sequentially continuous at* 0. *If X is a closed and convex subset of E, and $f\colon X \to X$ is a nonexpansive mapping, then the same conclusion of Theorem* 1 *holds true*.

PROOF. A Banach space E with a weakly sequentially continuous at 0 duality mapping satisfies Opial's condition, with $\leqslant$ instead of $<$ (see **[5]**). By uniform convexity of E we have that the equality can be verified in (1) only if $x^0 = x$ (see **[5]**). Then the hypotheses of our Theorem 1 are verified. Hence, its conclusion is true.

We observe that our starting point, i.e. Opial's condition, allows us to avoid the explicit use of a duality mapping and of convexity assumptions on E and X. Moreover, since in any separable Banach space we can introduce a new equivalent norm satisfying Opial's condition (see **[3]**), it is easy to show that our Theorem 1 can be used in a larger class of Banach spaces than Corollary 1, by considering separable, nonreflexive Banach spaces.

It would be interesting to give an example of a Banach space with Opial's condition, but for which any duality mapping is not weakly sequentially continuous at 0; unfortunately, I do not have such an example at present.

3. Corollaries. Now, we consider some corollaries of our Theorem 1 about weak convergence of the sequence $\{f^n(x)\}$.

The first is

COROLLARY 2. *If E, X, f are as in Theorem 1 and f is asymptotically regular, i.e. $f^{n+1}(x) - f^n(x) \xrightarrow{s} 0$, then $\{f^n(x)\}$ converges weakly to a fixed point of f, if $E(x) \neq \varnothing$.*

PROOF. Since $E(x) \neq \varnothing$, $\{f^n(x)\}$ is bounded. Let $y \in \omega_w(x)$. If $f^{n(j)}(x) \xrightarrow{w} y$, one has

$$\begin{aligned}
&\limsup_j \|f^{n(j)}(x) - f(y)\| \\
&\quad \leqslant \limsup_j \left\{\|f^{n(j)}(x) - f^{n(j)+1}(x)\| + c|f^{n(j)+1}(x) - f(y)|\right\} \\
&\quad \leqslant \limsup_j \|f^{n(j)}(x) - y\|
\end{aligned}$$

by which $y \in F(f)$ follows. Using Theorem 1 we conclude the proof.

In the following result we do not consider assumptions of asymptotic regularity on f.

COROLLARY 3. *Let E, X, f be as in Theorem 1. Let X be bounded and convex. If $\lambda \in]0,1[$ and $f_\lambda(x) = (1-\lambda)x + \lambda f(x)$, then $\{f_\lambda^n(x)\}$ converges weakly to a fixed point of f, if $F(f) \neq \varnothing$.*

PROOF. If X is bounded and convex, then $F(f) = F(f_\lambda) \neq \varnothing$; moreover, f_λ is asymptotically regular (see **[7]**). Then we can apply our Corollary 2 in order to obtain the result.

In **[12]** Opial obtained two similar theorems under the assumptions "E is uniformly convex" and "E has a weakly sequentially continuous duality mapping". These results are particular cases of our Corollaries 2 and 3 (see proof of Corollary 1).

Furthermore, the space l_1 satisfies Opial's condition, since strong and weak convergence in l_1 are the same; however, l_1 is not uniformly convex and does not have a duality mapping that is weakly sequentially continuous. Indeed, if a weakly sequentially continuous duality mapping exists, then the norm of l_1 has to be Gateaux differentiable (see **[4]**), whereas it is not Gateaux differentiable at the point $x = (1,0,0,\ldots)$. Hence, our results are strictly more general than the results of Opial.

The following result has been known only in uniformly convex Banach spaces with Opial's condition (see **[6]**); we observe that the theorem of Hirano is an extension of results due to Miyadera **[11]** and Baillon, Bruck and Reich **[1]**, who used a weakly sequentially continuous duality mapping in uniformly convex Banach spaces. For our purpose, we require the following definition **[2**, p. 56]: a Banach space E is said to be uniformly convex in every direction if, for each $z \in E$, $\|z\| = 1$, $M > 0$ and $\varepsilon > 0$, there is $\delta \in]0,1[$ for which $\|x\|$, $\|y\| \geqslant M$, $x - y = tz$ and $|t| \geqslant \varepsilon$ imply that

$$\tfrac{1}{2}\|x+y\| \leqslant (1-\delta)\max\{\|x\|, \|y\|\}.$$

Obviously, a uniformly convex Banach space is uniformly convex in every direction.

Now we are ready to give our result.

COROLLARY 4. *Let E, X, f be as in Theorem* 1. *We suppose that E is uniformly convex in every direction and $F(f) \neq \varnothing$. Then $\{f^n(x)\}$ converges weakly to a fixed point of f if and only if the sequence $\{f^n(x) - f^{n+1}(x)\}$ converges weakly to* 0.

PROOF. If $\{f^n(x)\}$ converges weakly to y, $y = f(y)$, we have the result trivially. Now, we prove that $f^n(x) - f^{n+1}(x) \xrightarrow{w} 0$ implies that $\{f^n(x)\}$ converges weakly to a fixed point of f.

Since $F(f) \neq \varnothing$, we can affirm that $\omega_w(x)$ is nonempty. Let y be a weak subsequential limit of $\{f^n(x)\}$; there is a subsequence $\{f^{n(j)}(x)\}$ which converges weakly to y; then, $\{f^{n(j)+p}(x)\}$ converges weakly to y for any $p \in N$. If $y \neq f(y)$, we put

$$r_p = \limsup_j \|f^{n(j)+p}(x) - y\|;$$

since $r_{p+1} \leqslant r_p$ for each $p \in N$, we have the existence of an $r \leqslant 0$ for which $r_p \to r$, $r_p \geqslant r$. Since $y \neq f(y)$, $r > 0$ easily follows.

Then, if $\sigma > 0$, $\bar{p} \in N$ exists for which

$$\text{(2)} \qquad \begin{aligned} \limsup_j \|f^{n(j)+p+1}(x) - f(y)\| &< r + \sigma, \\ \limsup_j \|f^{n(j)+p+1}(x) - y\| &< r + \sigma. \end{aligned}$$

Now, let $\delta \in]0,1[$ according to the definition of uniform convexity in every direction, with $M = r + \sigma$, $z = (y - f(y))/\|y - f(y)\|$, $t = \varepsilon = \|y - f(y)\|$.

We consider $\mu > 0$ such that $(1 - \delta)(r + \mu) < r$ and a $p' > \bar{p}$ for which (2) holds with μ instead of σ. By these inequalities there exists j' such that

$$\max\left\{\left\|f^{n(j)+p'+1}(x) - f(y)\right\|, \left\|f^{n(j)+p'+1}(x) - y\right\|\right\}$$
$$< \min(r + \mu, r + \sigma) \quad \text{for } j \geqslant j'.$$

And so by uniform convexity in every direction we have

$$\left\|f^{n(j)+p'+1}(x) - (y + f(y))/2\right\| \leqslant (1 - \delta)(r + \mu) \quad \text{for each } j \geqslant j';$$

this fact implies that

$$\begin{aligned} r_{p'+1} &= \limsup_j \left\|f^{n(j)+p'+1}(x) - y\right\| \\ &< \limsup_j \left\|f^{n(j)+p'+1}(x) - (y + f(y))/2\right\| \\ &\leqslant (1 - \delta)(r + \mu) \leqslant r, \end{aligned}$$

which contradicts the construction of r. Then, $y = f(y)$. Since (ii) of Theorem 1 is true, our result follows.

The following example shows that our Corollary 4 is strictly more general than the cited result by Hirano. We consider l_1 with the norm $|x| = (\|x\|_{l_1}^2 + \|x\|_{l_2}^2)^{1/2}$; it is known that l_1 endowed with this norm is uniformly convex in every direction (see [**13**, p. 18]). $(l_1, |\cdot|)$ is a good example for our purpose; but since, in l_1, strong and weak convergence coincide, our Corollary 4 in $(l_1, |\cdot|)$ is a particular case of our Corollary 2. Thus, we consider the following example of a Banach space E, such that

(a) E is uniformly convex in every direction with Opial's condition;

(b) E is not uniformly convex;

(c) in E, strong and weak convergence do not coincide, in order to prove that Corollary 4 is strictly more general than the result of [**6**] and it is "independent" from Corollary 2.

Let $E = l_1 \times l_2$, with the norm $|||(x, y)||| = (|x|^2 + \|y\|^2)^{1/2}$.

(c) is obviously true, since in l_2 the strong and weak convergence do not coincide; moreover, (b) is verified, since l_1 is not reflexive. In order to prove (a) we observe that E is uniformly convex in every direction, since in the contrary case an $(x, y) \neq 0$ and a bounded sequence $\{(x_n, y_n)\}$ exist in such a way that

$$2\left(|||(x_n, y_n) + (x, y)|||^2 + |||(x_n, y_n)|||^2\right) - |||2(x_n, y_n) + (x, y)|||^2 \to 0$$

(see [**13**, p. 8, (7)]); but, it is easy to prove that this contradicts uniform convexity in every direction of $(l_1, |\cdot|)$ or $(l_2, \|\cdot\|)$, using the definition of $\|\cdot\|$. Now, we prove that in E Opial's condition is verified. Let $\{(x_n, y_n)\}$ be a sequence in E such that $(x_n, y_n) \xrightarrow{w} (x, y) \in E$; we consider $(x', y') \neq (x, y)$. If $y \neq y'$, one has, since $x_n \xrightarrow{s} x$,

$$\begin{aligned}
\limsup_n |||(x_n, y_n) - (x, y)|||^2 &= \limsup_n \left\{|x_n - x|^2 + \|y_n - y\|^2\right\} \\
&\leqslant \limsup_n \|y_n - y\|^2 < \limsup_n \|y_n - y'\|^2 \\
&\leqslant \limsup_n \left\{|x_n - x'|^2 + \|y_n - y'\|^2\right\} \\
&= \limsup_n |||(x_n, y_n) - (x', y')|||^2;
\end{aligned}$$

if $x \neq x'$, then

$$\eta = \liminf_n \left\{\|x_n - x'\| : n \in N\right\} > 0.$$

Indeed, if $\eta = 0$, there is a subsequence $\{x_{k(n)}\}$ that converges strongly to x'. Since $\{x_n\}$ converges strongly to x, we have $x = x'$, a contradiction. Thus

$$\begin{aligned}
&\limsup_n |||(x_n, y_n) - (x, y)|||^2 = \limsup_n \left\{|x_n - x|^2 + \|y_n - y\|^2\right\} \\
&\leqslant \limsup_n \|y_n - y\|^2 \leqslant \limsup_n \|y_n - y'\|^2 \\
&< \limsup_n \left\{\eta + \|y_n - y'\|^2\right\} \leqslant \limsup_n \left\{|x_n - x'|^2 + \|y_n - y'\|^2\right\} \\
&= \limsup_n |||(x_n, y_n) - (x', y')|||^2,
\end{aligned}$$

and so the proof is complete. In passing, we observe that the above defined Banach space E does not have a weakly sequentially continuous mapping since its norm is not Gateaux differentiable at the point $(x, 0)$, where $x = (1, 0, 0, \ldots)$.

4. Semigroups of nonexpansive mappings. Let I be an unbounded subset of $[0, \infty)$ such that $t + h \in I$ for all $t, h \in I$, $t - h \in I$ and, for all $t, h \in I$, $t > h$ (e.g. $I = [0, \infty)$ or $I = N$). We consider a mapping S, from $I \times X$ into X, such that $S(t + h, x) = S(t, S(h, x))$ for all $t, h \in I$ and $x \in X$ and $\|S(t, x) - S(t, y)\| \leqslant \|x - y\|$ for all $t \in I$ and $x, y \in X$, i.e. S is a (not necessarily continuous) semigroup of nonexpansive mappings. Using standard proof as in Corollaries 2 and 4 we have

THEOREM 1′. *Let E, X be as in Theorem* 1 *and S, I as defined above. If S has a fixed point and $\{S(t + h, x) - S(t, x)\}_{t \geqslant 0}$ converges strongly to* 0, *then $\{S(t, x)\}_{t \geqslant 0}$ converges weakly to a fixed point of S. If, in addition, E is uniformly convex in every direction, then $\{S(t, x)\}_{t \geqslant 0}$ converges weakly to a fixed point of S, provided that S has a fixed point and that $\{S(t + h, x) - S(t, x)\}_{t \geqslant 0}$ converges weakly to* 0.

In this way we generalize two results due to Hirano [**6**] and Baillon, Bruck and Reich [**1**] obtained in uniformly convex Banach spaces with Opial's condition.

5. Mappings of asymptotically nonexpansive type. Following Kirk (see [**8**]) we sat that a continuous mapping $f\colon X \to X$, X a closed subset of a Banach space E, is said to be of asymptotically nonexpansive type if, for each $x \in X$

$$\limsup_{n} \left\{ \sup\left[\|f^n(x) - f^n(y)\| - \|x - y\| \right] : y \in X \right\} \leqslant 0.$$

The following result is the extension to the case of mappings of asymptotically nonexpansive type of a result of Miyadera (see [**11**]).

THEOREM 2. *Let E be a uniformly convex Banach space with Opial's condition and let X be a bounded and convex subset of E. If f is a mapping of asymptotically nonexpansive type, then $f^n(x) - f^{n+1}(x) \xrightarrow{w} 0$ implies that $\{f^n(x)\}$ converges weakly to a fixed point of f.*

PROOF. It is known that $F(f) \neq \varnothing$ [**8**]. Let $\{f^{n(j)}(x)\}$ and $y \in E$ be such that $f^{n(j)}(x) \xrightarrow{w} y$. If we put

$$r_p = \limsup_{j} \|f^{n(j)+p}(x) - y\|$$

and fix $\varepsilon > 0$, a $\bar{q} \in N$ exists for which $q > \bar{q}$ implies

$$\|f^{n(j)+p+q}(x) - f^q(y)\| \leqslant \varepsilon + \|f^{n(j)+p}(x) - y\| \quad \text{for each } p \in N,\ j \in N;$$

hence, $r_{p+q} \leqslant r_p + \varepsilon$. Since ε is an arbitrary number, an $r \geqslant 0$ exists in such a way that $r_p \to r$, $r_p \geqslant r$. Since in a uniformly convex Banach space, for any $\eta > 0$ there is $\delta \in]0, 1[$ such that

$$\tfrac{1}{2}\|x + y\| \leqslant (1 - \delta) \max\{\|x\|, \|y\|\} \quad \text{for any } x, y \in E$$

with $\|x - y\| \geqslant \eta \max\{\|x\|, \|y\|\}$, as in the proof of Corollary 4, we have that $y \in F(f)$, since $\{f^n(y)\}$ converges strongly to y. Since we can prove easily that the

$\lim_n \|f^n(x) - y\|$ exists for any $y \in F(f)$, using Opial's condition we have that $f^n(x) \xrightarrow{w} y$. The proof is complete.

Now, we consider the metric projection P, defined on X, with values in $F(f)$; for the sequence $\{P(f^n(x)\}$ we can obtain the following

THEOREM 3. *Let E, X, f, P be as above. Then, if $\{f^n(x)\}$ converges weakly to a $y \in F(f)$, the sequence $\{P(f^n(x))\}$ converges strongly to the same fixed point y.*

PROOF. At first we show that $\{\|f^n(x) - P(f^n(x))\|\}$ is a convergent sequence. If $r_q = \|f^q(x) - P(f^q(x))\|$, using a similar technique to that used in Theorem 2, we have that an $r \geqslant 0$ exists, with $r_q \to r$, $r_q \geqslant r$. If $r = 0$, it is easy to prove that $\{P(f^n(x))\}$ is a Cauchy sequence; when $r \neq 0$ and we suppose that $\{P(f^n(x))\}$ is not a Cauchy sequence, with a proof as in Corollary 4, we obtain the contradiction $r > r_q$ for a $q \in N$. Hence a $z \in E$ exists in such a way that $P(f^n(x)) \xrightarrow{s} z$. If $\{f^n(x)\}$ converges weakly to a $y \in F(f)$, we have only to prove that $z = y$. For this purpose we have

$$\begin{aligned} &\limsup_n \|f^n(x) - z\| \\ &\leqslant \limsup_n \{\|f^n(x) - P(f^n(x))\| + \|P(f^n(x) - z\|\} \\ &\leqslant \limsup_n \|f^n(x) - P(f^n(x))\| \leqslant \limsup_n \|f^n(x) - y\|; \end{aligned}$$

Opial's condition implies that $z = y$. The proof is complete.

REFERENCES

1. J. B. Baillon, R. E. Bruck and S. Reich, *On the asymptotic behavior of nonexpansive mappings and semigroups in Banach spaces*, Houston J. Math. **4** (1978), 1–9.

2. Cl. Crauthausen, G. Muller, J. Reinermann and R. Schoneberg, *New fixed point theorems for compact and nonexpansive mappings and applications to Hammerstein equations*, Sonderforschungsbereich 72, Approximation und Optimierung, Universitat Bonn, 1976.

3. D. van Dulst, *Equivalent norms and the fixed point property for nonexpansive mappings*, J. London Math. Soc. (2) **25** (1982), 139–144.

4. J. P. Gossez, *A note on multivalued monotone operators*, Michigan Math. J. **17** (1970), 347–350.

5. J. P. Gossez and E. Lami Dozo, *Some geometric properties related to the fixed point theory for nonexpansive mappings*, Pacific J. Math. **40** (1972), 565–573.

6. N. Hirano, *Nonlinear ergodic theorems and weak convergence theorems*, J. Math. Soc. Japan **34** (1982), 35–46.

7. S. Ishikawa, *Fixed points and iteration of a nonexpansive mapping in a Banach space*, Proc. Amer. Math. Soc. **59** (1976), 65–71.

8. W. A. Kirk, *Fixed point theorems for nonlipschitzian mappings of asymptotically nonexpansive type*, Israel J. Math. **17** (1974), 333–346.

9. K. Kobayashi, *On the asymptotic behavior of iterates of nonexpansive mappings in uniformly convex Banach spaces*, Proc. Japan Acad. Ser. A Math. Sci. **55** (1979), 209–212.

10. I. Miyadera, *Asymptotic behavior of iterates of nonexpansive mappings in Banach spaces*, Proc. Japan Acad. Ser. A Math. Sci. **54** (1978), 212–214.

11. ______, *Asymptotic behavior of iterates of nonexpansive mappings in Banach spaces*. II, Proc. Japan Acad. Ser. A Math. Sci. **54** (1978), 318–321.

12. Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591–597.

13. V. Zizler, *On some rotundity and smoothness properties of Banach spaces*, Dissertationes Math. (Rozprawy Mat.) **87** (1971), 1–33.

DIPARTIMENTO DI MATEMATICA, CITTÀ UNIVERSITARIA, CATANIA, ITALY