SUMMING GENERALIZED CLOSED U-SETS FOR WALSH SERIES

KAORU YONEDA

Abstract. A countable union of closed U-sets for Walsh series in certain generalized sense is again a U-set in the same sense.

1. Introduction. Let \( \mu \sim \sum_{k=0}^{\infty} \hat{\mu}(k)w_k(x) \) be a Walsh series. A subset \( E \) of the dyadic group is said to be a U-set if

\[
\sum_{k=0}^{\infty} \hat{\mu}(k)w_k(x) = 0 \quad \text{everywhere except on } E
\]

implies that \( \mu \) is the zero series.

Wade [2] proved that if \( E_1, E_2, \ldots \) are closed U-sets, then \( \bigcup_{k=1}^{\infty} E_k \) is also a U-set.

In this paper we shall generalize Wade's theorem. Let \( \mathcal{A} \) be a certain class of Walsh series. A subset \( E \) of the dyadic group is said to be a U-set for \( \mathcal{A} \) if \( \mu \in \mathcal{A} \) and

\[
\lim_{n \to \infty} \sum_{k=0}^{2^n-1} \hat{\mu}(k)w_k(x) = 0 \quad \text{everywhere except on } E
\]

imply that \( \mu \) is the zero series. We have already proved in [3] that when \( E \) is a closed subset of the dyadic group, (1) holds if and only if (2) holds and

\[
\hat{\mu}(k) = o(1) \quad \text{as } k \to \infty.
\]

Therefore a closed subset of the dyadic group is a U-set in the classical sense if and only if it is a U-set for the class of Walsh series \( \mu \) which satisfies (3).

When \( \mathcal{A} \) satisfies the following conditions, we say that \( \mathcal{A} \) satisfies the condition (L):

(i) a U-set for \( \mathcal{A} \) is of measure zero;

(ii) if \( \mu \in \mathcal{A} \), then

\[
\liminf_{n \to \infty} \left| \frac{1}{2^n} \sum_{k=0}^{2^n-1} \hat{\mu}(k)w_k(x) \right| = 0 \quad \text{everywhere};
\]

(iii) if \( \mu \) and \( \mu' \in \mathcal{A} \), then \( \alpha \mu + \alpha' \mu' \in \mathcal{A} \) for arbitrary real numbers \( \alpha \) and \( \alpha' \), where

\[
(\alpha \mu + \alpha' \mu') \sim \sum_{k=0}^{\infty} (\alpha \mu + \alpha' \mu')(k)w_k(x)
\]

\[
\equiv \sum_{k=0}^{\infty} (\alpha \hat{\mu}(k) + \alpha' \hat{\mu}'(k))w_k(x);
\]

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(iv) if $\mu \in \mathcal{A}$, then \[
\sum_{k=0}^{\infty} \hat{\mu}(k+j)w_k(x) \in \mathcal{A} \quad \text{for } j = 1, 2, \ldots.
\]

We shall prove the following theorem.

**Theorem 1.** When a class of Walsh series $\mathcal{A}$ satisfies the condition (L) and if $E_1, E_2, \ldots$ are closed $U$-sets for $\mathcal{A}$, then $\bigcup_{k=1}^{\infty} E_k$ is also a $U$-set for $\mathcal{A}$.

**2. Notations and lemmas.** In this paper we shall use the following notations. Let $I_p^n$ be the set of all 0-1 sequences, $(t_1, t_2, \ldots)$, such that $\sum_{k=1}^{n} t_k/2^k = p/2^n$. $I_p^n$ is called a *dyadic interval of rank* $n$. For convenience, $I_n(x)$ denotes the dyadic interval of rank $n$ containing $x$. A dyadic interval is closed and open. We refer the details of the dyadic group, Walsh functions, the operation $\cdot$ and so on to Fine’s paper [1].

**Lemma 2.** When $\mathcal{A}$ satisfies the condition (L), if $\mu \in \mathcal{A}$ and $I$ is a dyadic interval, then there exists a Walsh series $\mu^* \in \mathcal{A}$ which satisfies the following conditions:

(i) \[
\lim_{n \to \infty} \left| \sum_{k=0}^{2^n-1} \hat{\mu}(k)w_k(x) - \sum_{k=0}^{2^n-1} \hat{\mu}^*(k)w_k(x) \right| = 0 \quad \text{on } I,
\]

(ii) \[
\lim_{n \to \infty} \sum_{k=0}^{2^n-1} \hat{\mu}^*(k)w_k(x) = 0 \quad \text{uniformly everywhere except on } I.
\]

**Proof.** From the hypothesis, there exist an element of the dyadic group, $x_0$, and an integer $N$ such that $I = I_N(x_0)$. Set

$$
\hat{\mu}^*(k) = \frac{1}{2^N} \sum_{j=0}^{2^N-1} w_j(x_0)\hat{\mu}(k+j)
$$

for $k = 0, 1, \ldots$. Since

$$
\sum_{k=0}^{2^n-1} \hat{\mu}^*(k)w_k(x) = \sum_{s=0}^{2^n-1} \sum_{k=s2^N}^{(s+1)2^N-1} \hat{\mu}^*(k)w_k(x)
$$

$$
= \sum_{s=0}^{2^n-N-1} \left( \sum_{k=0}^{2^n-1} \hat{\mu}^*(s2^N+k) w_{s2^N+k}(x) \right)
$$

$$
= \sum_{s=0}^{2^n-N-1} \sum_{k=0}^{2^n-1} \sum_{j=0}^{2^n-1} w_j(x_0) \hat{\mu} \left( s2^N+k + j \right) w_{s2^N+k}(x)
$$

$$
= \sum_{s=0}^{2^n-N-1} \sum_{j=0}^{2^n-1} \frac{1}{2^N} w_j(x_0) \sum_{k=0}^{2^n-1} \hat{\mu} \left( s2^N+k + j \right) w_{s2^N+k}(x)
$$

$$
= \sum_{s=0}^{2^n-N-1} \sum_{j=0}^{2^n-1} \frac{1}{2^N} w_j(x_0) w_j(x) \sum_{k=0}^{2^n-1} \hat{\mu} \left( s2^N+k + j \right) w_{s2^N+k}(x) \times w_j(x)
$$

$$
= \sum_{s=0}^{2^n-N-1} \frac{1}{2^N} \left( \sum_{j=0}^{2^n-1} w_j(x_0 + x) \right) \left( \sum_{k=s2^N}^{(s+1)2^N-1} \hat{\mu}(k)w_k(x) \right)
$$

$$
= \left\{ \frac{1}{2^N} \sum_{j=0}^{2^n-1} w_j(x_0 + x) \right\} \left( \sum_{k=0}^{2^n-1} \hat{\mu}(k)w_k(x) \right),
$$

we have
and
\[ \sum_{j=0}^{2^N-1} w_j(x_0 + x) = \begin{cases} 2^N, & \text{for } x \in I_N(x_0), \\ 0, & \text{otherwise}, \end{cases} \]
we have
\[ \sum_{k=0}^{2^n-1} \mu^*(k) w_k(x) = \begin{cases} \sum_{k=0}^{2^n-1} \mu(k) w_k(x), & \text{for } x \in I_N(x_0), \\ 0, & \text{otherwise}. \end{cases} \]

It is obvious that \( \mu^* \) satisfies the conclusion.

**LEMMA 3.** If a Walsh series \( \mu \) satisfies the following conditions:

(i) \[ \lim_{n \to \infty} \inf \left| \sum_{k=0}^{2^n-1} \mu(k) w_k(x) \right| = 0 \text{ a.e.;} \]

(ii) \[ \sup_n \left| \sum_{k=0}^{2^n-1} \mu(k) w_k(x) \right| < \infty \text{ everywhere except on a countable set;} \]

(iii) \[ \lim_{n \to \infty} \inf \left| \frac{1}{2^n} \sum_{k=0}^{2^n-1} \mu(k) w_k(x) \right| = 0 \text{ everywhere;} \]

then \( \mu \) is the zero series.

Lemma 3 is Theorem 3 in [3].

**COROLLARY 4.** When \( \mathcal{A} \) satisfies the condition (L), if \( E \) is a closed \( U \)-set for \( \mathcal{A} \) and \( I \) is a dyadic interval which contains \( E \), if a Walsh series \( \mu \in \mathcal{A} \) satisfies the following conditions:

(i) \[ \sup_n \left| \sum_{k=0}^{2^n-1} \mu(k) w_k(x) \right| < \infty \text{ everywhere on } I \setminus E; \]

(ii) \[ \lim_{n \to \infty} \inf \left| \sum_{k=0}^{2^n-1} \mu(k) w_k(x) \right| = 0 \text{ a.e. on } I; \]

then
\[ \lim_{n \to \infty} \sum_{k=0}^{2^n-1} \mu(k) w_k(x) = 0 \text{ everywhere on } E. \]

**PROOF.** Since \( E \) is a closed set, for \( x_0 \in I \setminus E \), there exists an integer \( N \) such that \( I_N(x_0) \subset I \) and
\[ I_N(x_0) \cap E = \emptyset. \]

Let \( \mu^* \) be a Walsh series which is introduced in Lemma 2. Hence \( \mu^* \) satisfies that
\[ \lim_{n \to \infty} \sum_{k=0}^{2^n-1} \mu^*(k) w_k(x) = 0 \text{ everywhere on } I_N^*(x_0). \]
We shall prove that \( \mu^* \) satisfies the conditions of Lemma 3. Since \( \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x) \) and \( \sum_{k=0}^{2^n-1} \hat{\mu}^*(k) w_k(x) \) are equiconvergent on \( I_N(x_0) \), \( \mu^* \) satisfies (5), (i) and (ii) of Lemma 3. From (i) we have

\[
\sup_n \left| \sum_{k=0}^{2^n-1} \hat{\mu}^*(k) w_k(x) \right| < \infty \quad \text{everywhere on } I_N(x_0).
\]

On the other hand from (5), (6) holds on \( I_N(x_0) \). Hence (6) holds everywhere. From the definition of \( \hat{\mu}^*(k) \) and the hypothesis, we have \( \mu^* \in \mathcal{A} \). By Lemma 3, \( \mu^* \) is the zero series. Then, we have

\[
\lim_{n \to \infty} \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x) = 0 \quad \text{everywhere on } I \setminus E.
\]

Let \( \mu^{**} \) be a Walsh series associated with \( I \) which is introduced in Lemma 2. Since \( \mu^{**} \in \mathcal{A} \) and the \( 2^n \)-th partial sums of \( \mu^{**} \) and \( \mu \) are equiconvergent on \( I \), we have

\[
\lim_{n \to \infty} \sum_{k=0}^{2^n-1} \hat{\mu}^{**}(k) w_k(x) = 0 \quad \text{everywhere } I \setminus E.
\]

On the other hand, (7) holds on \( I^c \). Hence (7) holds everywhere except on \( E \). Since \( E \) is a \( U \)-set for \( \mathcal{A} \), \( \mu^{**} \) is the zero series. Therefore, (7) holds everywhere on \( I \). Since the \( 2^n \)-th partial sums of \( \mu \) and \( \mu^{**} \) are equiconvergent, the proof of Corollary 4 is complete.

**Lemma 5.** Let \( f_n, n = 0, 1, \ldots, \) be a function which is continuous on the dyadic group, then the following set

\[
N = \left\{ x: \limsup_{n \to \infty} |f_n(x)| = \infty \right\}
\]

is empty, countable or of the second category on itself.

The proof is due to [2].

**3. Proof of Theorem 1.** Set \( E = \bigcup_{i=1}^{\infty} E_i \). Let \( \mu \) satisfy

\[
\liminf_{n \to \infty} \left| \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x) \right| = 0 \quad \text{everywhere except on } E.
\]

Each \( E_i \) is a \( U \)-set for \( \mathcal{A} \), then from (i) of (L), \( E_i \) is of measure zero. Hence \( E \) is of measure zero. Consequently \( \mu \) satisfies (8) a.e. Set

\[
N = \left\{ x: \limsup_{n \to \infty} \left| \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x) \right| = \infty \right\}.
\]

Since \( \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x) \) is a continuous function on the dyadic group, by Lemma 5, three cases arise.

Now we shall assume that \( N \) is of the second category on itself. Set \( N_i = N \cap E_i \). Then, there exist a dyadic interval \( I \) and an integer \( i_0 \) such that \( N \cap I \neq \emptyset \) and \( N_{i_0} \cap I \) is dense in \( N \cap I \). Since \( E_{i_0} \) is closed, we have \( N_{i_0} = N \cap E_{i_0} \). We shall prove that

\[
N \cap I = E_{i_0} \cap N \cap I \subseteq E_{i_0} \cap I.
\]
It is obvious that \( N \cap I \supseteq E_{i_0} \cap N \cap I \). If \( x \in N \cap I \), then there exists a sequence of elements \( \{x_k\}_k \) such that \( x_k \in N_{i_0} \cap I \) and \( \lim_{k \to \infty} x_k = x \). Since \( x_k \in N_{i_0} \) and \( x_k \in I \), we have \( x_k \in E_{i_0} \), \( E_{i_0} \) is closed, therefore we have \( x \in E_{i_0} \). Hence we proved the conclusion. It is obvious that \( E_{i_0} \cap I \) is a closed \( U \)-set for \( A \) and that \( (E_{i_0} \cap I) = I \setminus E_{i_0} \).

Assume that \( x \notin E_{i_0} \cap I \). Then \( x \notin N \cap I \). Hence we have

\[
\sup_n \left| \sum_{k=0}^{2^n-1} \mu(k)w_k(x) \right| < \infty \quad \text{in } I \setminus (E_{i_0} \cap I) = I \setminus E_{i_0}.
\]

By Corollary 4, we have

\[
\lim_{n \to \infty} \sum_{k=0}^{2^n-1} \mu(k)w_k(x) = 0 \quad \text{everywhere on } I.
\]

Hence we proved that

\[
N \cap I = \emptyset.
\]

(10) contradicts the assumption \( N \cap I \neq \emptyset \). Therefore we have that \( N \) is not of the second category on itself. The proof is complete.

A subset \( E \) of the dyadic group is said to be a \( U_1 \)-set for \( A \) [3] if \( \mu \in A \) and

\[
\liminf_{n \to \infty} \left| \sum_{k=0}^{2^n-1} \mu(k)w_k(x) \right| = 0 \quad \text{everywhere except on } E
\]

imply that \( \mu \) is the zero series.

We can prove analogously to Theorem 1 the following theorem.

**Theorem 1'**. When a class of Walsh series \( A \) satisfies the condition (L), if \( E_1, E_2, \ldots \) are closed \( U_1 \)-sets for \( A \), then \( \bigcup_{k=1}^{\infty} E_k \) is also a \( U_1 \)-set for \( A \).

**References**


Department of Mathematics, University of Osaka Prefecture, Sakai, Osaka, Japan