ON THE SELFADJOINTNESS OF DIRAC OPERATORS WITH ANOMALOUS MAGNETIC MOMENT

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Abstract. We provide a new proof of Behncke's remarkable result that the Coulombic Dirac equation with nonzero anomalous magnetic moment is essentially selfadjoint (on $C_0^\infty(\mathbb{R}^3)^4$) for any value of the Coulomb charge.

In this note we shall consider Dirac operators. In the simplest version, these have the form

\begin{equation}
H = \bar{\alpha} \cdot \vec{p} + m\beta + V
\end{equation}

where $\vec{p} = -i \vec{\nabla}$ on $L^2(\mathbb{R}^3)$ and $H$ acts on $L^2(\mathbb{R}^3, d^3x; C^4)$. $\alpha, \beta$ are $4 \times 4$ matrices, written in terms of the conventional $2 \times 2$ Pauli sigma matrices, $\sigma$, as $2 \times 2$ blocks of $2 \times 2$ matrices:

$$
\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \bar{\alpha} = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}.
$$

It is well known (see e.g. [1, 8, 9]) that for $V = e|\vec{x}|^{-1}$, (1) is essentially selfadjoint on $C_0^\infty(\mathbb{R}^3)^4$ if and only if $e \leq \sqrt{3}/4$, and strange spectral properties occur if $e > 1$ [11]. Indeed, it has been speculated that these difficulties have physical significance for the stability of the world if superheavy nuclei with charge $Z > 137$ exist (written back in conventional units $Z = e\alpha^{-1}$ with $\alpha$ the fine structure constant); see [6, 10] and the references therein. We feel that these speculations are ill founded for a number of reasons including the theme of this note.

Equation (1) corresponds to the equation for an electron with magnetic moment 1 (in units of Bohr magnetons), but it is known that the actual value is $1 + \mu$ where $\mu = 0.001159$ is called the anomalous magnetic moment [5] (understood from the point of view of quantum electrodynamics). Equation (1) in the presence of such an anomalous moment should be replaced by [13]

\begin{equation}
H = \bar{\alpha} \cdot \vec{p} + m\beta + V - \frac{\mu}{2m} \vec{\tau} \cdot \vec{\nabla} V
\end{equation}

with

$$
\vec{\tau} = \begin{pmatrix} 0 & i\sigma \\ -i\sigma & 0 \end{pmatrix}.
$$

\begin{flushright}
Received by the editors June 12, 1984.
1980 Mathematics Subject Classification. Primary 81C10, 35F10; Secondary 34B20, 47F05.
\textsuperscript{1}Research partially supported by USNSF Grant MCS-81-20833.

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0002-9939/85 $1.00 + $.25 per page
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Recently, Behncke [3] (see also his papers [4, 16] and the earlier work of Barut and Kraus [2]), discovered the remarkable result that if $\mu \neq 0$ and $V = e|\xi|^{-1}$, then (2) is essentially selfadjoint on $C_0^\infty(\mathbb{R}^3)^4$ (here $C_0^\infty = C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ for any $e \neq 0$).

Behncke's analysis depends essentially on the central symmetry of $V(x) = e|\xi|^{-1}$ which allows one to write (2) as a direct sum of (two-component) ODE's. He analyzes these new operators by the well-developed selfadjointness techniques of such ODE's (see e.g. Weidman [15]). Our goal here is to prove Behncke's result using operator theoretic methods.

Absorbing $\mu/2m$ into $V$, (2) finally becomes

$$(2') H = \bar{a} \cdot \bar{p} + m\beta + \frac{2m}{\mu} V - \bar{\tau} \cdot \nabla V.$$

In order to study (2'), it turns out to be useful to introduce operators of the type

$$\begin{pmatrix} 0 & A_- \\ A_+ & 0 \end{pmatrix}, \quad S = \begin{pmatrix} S_+ & 0 \\ 0 & S_- \end{pmatrix}.$$  

We first state (see Kato [7] for discussions of $A$-boundedness)

**Proposition 1.** $S$ is $C$-bounded with relative bound zero if and only if $S_+$ is $A_+$-bounded and $S_-$ is $A_-$-bounded, each with relative bound zero.

**Proof.** $\|C(\varphi_+, \varphi_-)\|^2 = \|A_+\varphi_+\|^2 + \|A_-\varphi_-\|^2$ and $\|S(\varphi_+, \varphi_-)\|^2 = \|S_+\varphi_+\|^2 + \|S_-\varphi_-\|^2$. $\square$

This leaves the question of essential selfadjointness of operators of the form $C$.

**Proposition 2.** An operator of the form $C$ is essentially selfadjoint if and only if

(i) $(A_+)^* = \overline{A_+},$

(ii) $(A_-)^* = \overline{A_-}.$

Moreover, (i) is equivalent to (ii).

**Proof.** The equivalence of (i) and (ii) follows from $X^{**} = \overline{X}$ and $\overline{X}^* = X^*$. The first assertion follows from the observation that $\overline{C} = (\begin{smallmatrix} 0 & A_- \\ A_+ & 0 \end{smallmatrix})$ and secondly from the easy calculation that $\overline{C}^* = (\begin{smallmatrix} 0 & A^* \\ A^* & 0 \end{smallmatrix})$. $\square$

**Remark.** Nelson has noticed [12] that the fact that $C = (\begin{smallmatrix} 0 & A \\ A & 0 \end{smallmatrix})$ is selfadjoint if $A$ is closed provides a trivial proof of von Neumann's theorem that $A^*A$ is densely defined as selfadjoint: Just notice that by the spectral theorem, if $C$ is self-adjoint, then $C^2$ is densely defined and selfadjoint.

The final abstract result that we require is:

**Proposition 3.** Let $A, B$ be operators so that $B \subset A^*$, $A[D(A)] \subset D(\overline{B})$ and $BA$ is essentially selfadjoint on $D(A)$. Then $\overline{B} = A^*.$

**Proof.** By the last proposition, it is sufficient to prove $C = (\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix})$ is essentially selfadjoint on $D(B) \oplus D(A)$. Now $C^* = (\begin{smallmatrix} 0^* & B^* \\ A^* & 0 \end{smallmatrix})$ on $D(A^*) \oplus D(B^*)$. Suppose that $(u, v) \in D(A^*) \oplus D(B^*)$ with $A^*u = \pm iv, B^*v = \pm iu$. Let $\varphi \in D(A)$. Then

$$(\overline{BA}\varphi, v) = (A\varphi, B^*v) = \pm i(A\varphi, u) = \pm i(\varphi, A^*u) = -(\varphi, v)$$
so \( v \perp (\overline{BA} + 1)[D(A)] \), violating the assumed essential selfadjointness of the positive operator \( \overline{BA} \). Thus \( v \) and so \( u \) equal zero. It follows that \( C \) has zero deficiency indices so it is essentially selfadjoint. □

The standard angular momentum decomposition of Dirac operators when \( V(x) = V(|x|) \), shows that the operator of (2') is unitarily equivalent (under a transformation taking \( C_{00}^\infty(R^3) \) to \( C_{0}^\infty(0, \infty) \)) to a direct sum of operators \( H_{j,\sigma} \) indexed by \( j = \frac{1}{2}, \frac{3}{2}, \ldots \) and a sign \( \sigma = \pm \) (and with the space indexed by \( j \) occurring \( 2j + 1 \) times) with \( H_{j,\sigma} \) acting on \( L^2((0, \infty), dr; C^2) \) by [2, 3]

\[
H_{j,\sigma} = \begin{pmatrix}
S_{+,j,\sigma} & A_{-,j,\sigma} \\
A_{+,j,\sigma} & S_{-,j,\sigma}
\end{pmatrix},
\]

where \( S_{\pm} = 2mV(r)/\mu \pm m \) and

\[
A_{\pm} = \pm \frac{d}{dr} - \frac{\kappa(j, \sigma)}{r} - \frac{dV}{dr}
\]

where \( \kappa(j, \sigma) = \sigma(j + 1/2) \). Define

\[
W(r) = -\frac{\kappa(j, \sigma)}{r} - \frac{dV}{dr}.
\]

Then, on \( C_{0}^\infty(0, \infty) \), \( A_{+}^* \supset A_{-} \) and

\[
B_{\pm} = A_{\pm}^* A_{\pm} = -\frac{d^2}{dr^2} + W_{\pm}^2 \equiv \frac{-d^2}{dr^2} + V_{\text{eff}}^\pm
\]

where \( V_{\text{eff}}^\pm = (V')^2 + 2\kappa r^{-1}V' \mp V'' - (\kappa^2 \pm \kappa)r^{-2} \). If \( V = e|x|^{-1} \), then \( W \sim e|x|^{-2} \) for small \( |x| \) and

\[
V_{\text{eff}}^\pm \sim e^2|x|^{-4}
\]

for small \( |x| \); so (6) is essentially selfadjoint on \( C_{0}^\infty(0, \infty) \) by well-known results (the usual argument is one-dimensional—see Reed-Simon [14, Theorem X.10], but there are multidimensional arguments which apply also—see Reed-Simon [14, Theorem X.30]). Moreover, by (7)

\[
V^2 \leq eV_{\text{eff}}^\pm + C(\epsilon)
\]

so \( V^2 \) is a form bounded perturbation of \( B_{\pm} \). Thus, by Proposition 1.3, we have proven Behncke’s theorem:

**Theorem 5.** If \( \mu \neq 0 \) and \( V = e|x|^{-1} (\epsilon \neq 0) \), then (2) is essentially selfadjoint on \( C_{0}^\infty(R^3)^4 \).

The difficulties with noncentral potentials is shown by the fact that while (7), (8) hold for each value of \( j \), they are not uniform in \( j \).

We end by noting that one can compare our proof with that of Behncke [3] by noting the conditions under which our respective arguments apply. Behncke requires:

- (B1) \( V, V' \in L^2_{\text{loc}}(0, \infty), \mu \neq 0; \)
- (B2) Sgn \( V' \) constant near 0;
Our proof requires (if one uses Wüst’s Theorem [14, Theorem X.14] in borderline cases to add on \( S \))

\[(\text{GST1}) \ V'' \in L^2_{\text{loc}}(0, \infty), \mu \neq 0,\]

\[(\text{GST2}) \ V_{\text{eff}}(r) \geq 3/4r^2 - d \text{ for some } d \text{ and } r \text{ small or the same result for } V_{\text{eff}},\]

\[(\text{GST3}) \ V_J(r) - V_2(r) \geq -(4r^{-2}) - d \text{ for some } d \text{ where } V_{\text{eff}} \text{ are defined by (5), (6), i.e.,}\]

\[V_{\text{eff}}^+ = (V')^2 + 2\kappa r^{-1}V' \mp V'' + (\kappa^2 \pm \kappa)r^{-2}.\]

\( V'' \in L^2_{\text{loc}} \) in (GST1) can be replaced by \( V'' \in L^1_{\text{loc}}(0, \infty) \) using standard techniques.

Since Behncke has no conditions on \( V'' \), his results are, in a sense, stronger. But curiously, he does not allow \( V = 0 \) or \( V = [\ln(|r|^{-1} + 2)]^\alpha \) with \( \alpha < 1 \) while we do.

One of us (F. Gesztesy) would like to thank R. Vogt, W. Luxemburg and B. Simon for the hospitality of Caltech during a visit in April 1983.

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