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A SOLUTION OF ULAM'S PROBLEM ON RELATIVE MEASURE

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ABSTRACT. Suppose $\mathscr{A}$ is a collection of subsets of the unit interval and, for $A \in \mathscr{A}$, μ_A is a Borel measure on A which vanishes on points and gives A measure 1. The system μ_A ($A \in \mathscr{A}$) is called a *coherent system* if $\mu_A(C) = \mu_A(B)\mu_B(C)$ whenever $A \supseteq B \supseteq C$ are in $\mathscr{A}$ and all terms are defined. The existence of a coherent system for the collection of perfect sets is shown to be independent of Zermelo-Fraenkel set theory with the axiom of dependent choices.

In [**4**, problem 7, pp. 77–78] Ulam asked how large $\mathscr{A}$ can be if it is the domain of a coherent system. Assuming the axiom of choice, $\mathscr{A}$ can be all sets which are not ruled out by some obviously necessary restriction; for example, $\mathscr{A}$ cannot contain any countable sets. By restricting Lebesgue measure, a coherent system for the collection of sets of positive Lebesgue measure can be constructed (without using the axiom of choice). In particular, there is a coherent system on the collection of open sets. Without assuming the axiom of choice, the most natural question is whether a coherent system exists for the collection of perfect sets. The main theorem of this paper states that, assuming the principle of dependent choice, rather than the full axiom of choice, one cannot prove the existence of a coherent system for the collection of perfect sets. The proof of the main theorem also shows that the usual axioms for set theory, including the axiom of choice, are not sufficient to prove there is a definable coherent system on the collection of perfect sets.

1. Introduction. Most of the notation in this paper is standard, but some unusual liberties, which I will now list, are taken. For unfamiliar notation not discussed here, consult [**2**].

I will usually work with the Cantor space, rather than the unit interval, and will use the customary representation of it as ${}^{\omega}2$ with the product topology, where 2 is given the discrete topology. $2^{<\omega}$ is the collection of all finite sequences of 0's and 1's. A *perfect tree* is a nonempty subset A of $2^{<\omega}$ such that: (1) if $\eta \in A$ and τ is an initial segment of η, then $\tau \in A$; (2) if $\eta \in A$ then there are extensions η_1 and η_2 of η in A which disagree at some point. There is a natural correspondence between perfect subsets of ${}^{\omega}2$ and perfect trees, and I will often identify a perfect set with its tree. By a *finite tree* I will always mean a finite subset t of $2^{<\omega}$ which satisfies (1) and

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which has the additional property that if n is the height of t (i.e. n is the length of the longest element of t) then every element of t has an extension in t of length n. The elements of t of length n are called *maximal nodes*. If t and s are finite trees, then t is an *end extension* of s if $s \subseteq t$ and every element of $t - s$ extends a maximal node of s. If T is either a finite tree or a perfect tree and $\eta \in T$, then $T \upharpoonright \eta$ is defined to be $\{\tau \in T\colon \text{either } \tau \subseteq \eta \text{ or } \eta \subseteq \tau\}$.

Suppose A is a perfect tree and μ is a Borel measure on A, i.e. on the perfect subset of the Cantor space corresponding to A. If $t \subseteq A$ is a finite tree, then t determines a clopen subset U of the corresponding perfect set. I will write $\mu(t)$ for $\mu(U)$. As a special case, $\mu(\eta)$ is $\mu(\{x \in {}^{\omega}2\colon \eta \subseteq x\})$ if $\eta \in A$. Also, if B is a perfect tree and $B \subseteq A$, I will write $\mu(B)$ for the measure of the perfect set corresponding to B. So $\mu \upharpoonright A$ is the function with domain A with $(\mu \upharpoonright A)(\eta) = \mu(\eta)$. A well-known fact is that μ is completely determined by $\mu \upharpoonright A$. In other words, if ν is a Borel measure on A with $\nu \upharpoonright A = \mu \upharpoonright A$, then $\nu = \mu$. This can be shown withoput using the Axiom of Choice.

Finally, a few words on forcing notation. If M is a model of ZFC and P is a poset in M, I will write M^P for $M^{\mathscr{B}}$, where $\mathscr{B}$ is the Boolean algebra of regular open subsets of P. Elements of M^P will be denoted by underlined symbols with one exception. If $x \in M$ there is a canonical name for x in M^P; I will abuse notation and use x for its own canonical name. I will sometimes neglect the superscript on objects constructed in M. This happens for example in §2 where, after defining a particular poset $Q^*Q(\underline{\mu}, r)$, I continue to write $Q^*Q(\underline{\mu}, r)$ for $(Q^*Q(\underline{\mu}, r))^M$. $M^P \models \varphi$ abbreviates $p \Vdash \varphi$ for all $p \in P$.

2. Preliminaries. This section contains the basic lemmas and definitions to be used in the next section in the proof of the main theorem.

DEFINITION 1. A poset P is *homogeneous* if for all $p, q \in P$ there is an automorphism φ of P such that $\varphi(p)$ and q are compatible.

The next lemma is well known (see exercise 25.9 in Jech [**2**]).

LEMMA 1. *Assume M is a model of* ZFC *and P is a homogeneous poset in the sense of M. If $\varphi(v_1,\dots,v_n)$ is a formula and $x_1,\dots,x_n \in M$, then either $M^P \models \varphi(x_1,\dots,x_n)$ or $M^P \models \neg\varphi(x_1,\dots,x_n)$.*

LEMMA 2. *Assume M is a transitive model of* ZFC, *κ is an uncountable cardinal in the sense of M, and P is the poset for adding κ Cohen reals. If $G \subseteq P$ is M-generic and s is a countable sequence of ordinals in $M[G]$, then $M[G] = M[s][H]$ for some $H \subseteq P$ which is $M[s]$-generic.*

SKETCH OF PROOF. There is an $X \subseteq \kappa$ which is countable in the sense of M such that $s \in M[G \cap P_X]$, where $P_X = \{p \in P\colon \text{domain}(p) \subseteq X\}$. If $M[s] = M[G \cap P_X]$, then $M[G]$ is a generic extension of $M[s]$, via $P_{\kappa - X}$, which is isomorphic to P. Otherwise, $M[G \cap P_X]$ is a generic extension of $M[s]$ by a single Cohen real, and $M[G]$ is a generic extension of $M[s]$ by a single Cohen real followed by a generic extension, via $P_{\kappa - X}$, which amounts to a generic extension via P. □

DEFINITION 2. Q is the poset which adds a perfect tree with finite conditions: the conditions of Q are finite trees which are ordered by end extension.

DEFINITION 3. Assume A is a perfect tree, μ is a Borel measure on A, and r is a rational number. $Q(\mu, r)$ is the poset which adds a perfect subtree of A of measure r with finite conditions: the conditions are the finite trees t, with $\mu(t) > r$, which are ordered by end extension.

Recall that if P is a countable poset then a Cohen real will add a generic filter for P; for example, a Cohen real will add generic filters for Q and $Q(\mu, r)$.

LEMMA 3. *Assume M is a transitive model of* ZFC *and $\underline{A}$ is the name in M for the perfect tree which Q adds. If $\underline{\mu} \in M^Q$, $M^Q \models$ "$\underline{\mu}$ is a Borel measure on $\underline{A}$", and r is a rational number, then*

(1) $\{(t, s): t, s \in Q, s \subseteq t, t \Vdash$ "$\underline{\mu}(s) > r$" *and t and s have the same height*$\}$ *is dense in $Q^*Q(\underline{\mu}, r)$; and*

(2) $M^{Q^*Q(\underline{\mu}, r)} \models$ "*$\underline{B}$ is M-generic for Q*", *where $\underline{B}$ is the name for the perfect tree which $Q(\underline{\mu}, r)$ adds.*

PROOF. Let R be the set of conditions from (1). I will only prove (2). Assume $D \in M$ is dense in Q. The following claim suffices.

Claim. $\{(t, s) \in R: s \in D\}$ is dense in $Q^*Q(\underline{\mu}, r)$.

Assume $(u, v) \in R$. Choose $s \leqslant v$ with $s \in D$ and let $t = \{\eta \in 2^{\leqslant n}: \eta$ extends a maximal node of u, and if η extends a maximal node of v then $\eta \in s\} \cup u$, where n is the height of s. Note that $(t, s) \in R$ since $t \Vdash$ "$\underline{\mu}(s) = \underline{\mu}(v) > r$". □

3. The main theorem.

THEOREM. *If* ZF *is consistent, so is* ZF + DC + "*there is no coherent system for the collection of perfect sets*".

The rest of this section is devoted to proving the theorem.

Assume M is a transitive model of ZFC, κ is an uncountable cardinal in the sense of M, P is the poset for adding κ Cohen reals, and $G \subseteq P$ is M-generic. Let S be the collection of sequences of ordinals of length ω in $M[G]$ and define N to be $\mathrm{HOD}(S)^{M[G]}$. As in [3], N is a model of ZF + DC and ${}^{\omega}N \cap M[G] \subseteq N$. Let $\mathscr{A}$ be the collection of perfect trees in N. I will show that in N there is no coherent system on $\mathscr{A}$ from which the theorem follows.

Notice that $\mathscr{A}$ is also the set of perfect trees in $M[G]$ and any set of reals in $M[G]$ which is Borel in $M[G]$ is also in N and Borel in N.

Claim 1. *If $A \in \mathscr{A}$, $\mu \in M[G]$ and $M[G] \models$ "μ is a Borel measure on A", then $\mu \in N$.*

PROOF. μ can be reconstructed from $\mu \restriction A$ which is in N. □

The statement "μ_A $(A \in \mathscr{A})$ is a coherent system" is absolute between N and $M[G]$; so, by Lemma 1, it suffices to show there is no coherent system μ_A $(A \in \mathscr{A})$ in the sense of $M[G]$ which is in $\mathrm{OD}(S)^{M[G]}$.

The rest of the proof is by contradiction. Assume that μ_A $(A \in \mathscr{A})$ is coherent in the sense of $M[G]$ and in $\mathrm{OD}(S)^{M[G]}$. Let $\Phi(x, y, z)$ be a formula and $\vec{s} \in S$ such that $\mu = \mu_A$ iff $M[G] \models \Phi(A, \mu, \vec{s})$. By Lemma 2 of §1, I may assume that $\vec{s} \in M$ (otherwise conclude the argument with M replaced by $M[\vec{s}]$).

Claim 2. *If* $A \in \mathscr{A}$ *then* $\mu_A \upharpoonright A \in M[A]$.

PROOF. Let $H \subseteq P$ be $M[A]$-generic, with $M[G] = M[A][H]$ by Lemma 2 of §1. If $\eta \in A$, p is rational, and $\mu_A(\eta) \leqslant p$, then $M[A]^P \models$ "if μ is unique such that $\Phi(A, \mu, \vec{s})$, then $\mu(\eta) \leqslant p$". So, $\mu_A \upharpoonright A$ can be defined in $M[A]$ by $\mu_A(\eta) = \inf\{ p \in \mathbf{Q}$: $M[A]^P \models$ "if μ is unique such that $\Phi(A, \mu, \vec{s})$, then $\mu(\eta) \leqslant p$"$\}$. □

Claim 3. *There is a name* $\underset{\sim}{\nu} \in M^Q$ *such that, whenever* $A \in M[G]$ *is M-generic for* Q, $\mu_A \upharpoonright A$ *is the interpretation of* $\underset{\sim}{\nu}$ *using the M-generic filter on Q associated with A.*

PROOF. I will show $M^Q \models$ "there exists ν such that $M[\underset{\sim}{A}]^P \models$ 'if $\Phi(\underset{\sim}{A}, \mu, \vec{s})$ then $\mu \upharpoonright \underset{\sim}{A} = \nu$' ", where $\underset{\sim}{A}$ is the name for the perfect tree which Q adds. This is done by showing that the collection of conditions which force this statement is dense.

Suppose $t \in Q$. There is an M-generic filter J in $M[G]$ which contains t (see the remark in §2 after Definition 3). Let A be the corresponding perfect tree, i.e., the union of the filter. By Claim 2, $\mu_A \upharpoonright A \in M[A]$. Let $H \subseteq P$ be $M[A]$-generic with $M[A][H] = M[G]$. $M[A][H] \models$ "if $\Phi(A, \mu, \vec{s})$ then $\mu \upharpoonright A = \nu$", where $\nu = \mu \upharpoonright A$. Therefore $M[A] \models$ "there exists ν such that, for some $p \in P$, $p \models$ 'if $\Phi(A, \mu, \vec{s})$ then $\mu \upharpoonright A = \nu$' ". By the homogeneity of P, $M[A] \models$ "there exists ν such that $M[A]^P \models$ 'if $\Phi(A, \mu, \vec{s})$ then $\mu \upharpoonright A = \nu$' ". There is a condition $s \in J$ such that $s \Vdash$ "there exists ν such that $M[\underset{\sim}{A}]^P \models$ 'if $\Phi(\underset{\sim}{A}, \mu, \vec{s})$ then $\mu \upharpoonright \underset{\sim}{A} = \nu$' ". s is compatible with t since both are in J. □

Let $\underset{\sim}{\mu} \in M^Q$ be the measure on $\underset{\sim}{A}$ generated by $\underset{\sim}{\nu}$ in M^Q, and let $\underset{\sim}{B}_r$ be the name for the perfect tree added by $Q(\underset{\sim}{\mu}, r)$ in $M^{Q*Q(\mu, r)}$.

Recall from Lemma 3 of §2 that $R_r = \{(t, s)$: $t, s \in Q$, $s \subseteq t$, $t \Vdash$ "$\mu(s) > r$" and s has the same height as $t\}$ is dense in $Q*Q(\underset{\sim}{\mu}, r)$.

Claim 4. *Assume* $t, s \in Q$ *have the same height*, $s \subseteq t$, η *is the stem of* s, *and* $\eta^\frown 0$ *and* $\eta^\frown 1$ *are in* s. *If* $t \Vdash$ "$\underset{\sim}{\mu}(s \upharpoonright \eta^\frown 0) < \underset{\sim}{\mu}(s \upharpoonright \eta^\frown 1)$", *then* $s \Vdash$ "$\underset{\sim}{\mu}(s \upharpoonright \eta^\frown 0) < \underset{\sim}{\mu}(s \upharpoonright \eta^\frown 1)$".

PROOF. Assume $t \Vdash$ "$\underset{\sim}{\mu}(s \upharpoonright \eta^\frown 0) < \underset{\sim}{\mu}(s \upharpoonright \eta^\frown 1)$" and $s \nVdash$ "$\underset{\sim}{\mu}(s \upharpoonright \eta^\frown 0) < \underset{\sim}{\mu}(s \upharpoonright \eta^\frown 1)$" in order to reach a contradiction.

Choose $s_1 \leqslant s$ such that $s_1 \Vdash$ "$\underset{\sim}{\mu}(s \upharpoonright \eta^\frown 0) \geqslant \underset{\sim}{\mu}(s \upharpoonright \eta^\frown 1)$", and let $t_1 = t \cup \{\tau \in 2^{\leqslant n}$: τ extends a maximal node of t, and if τ extends a maximal node of s then $\tau \in s \supsetneqq 1\}$, where n is the height of s_1. Both s_1 and t_1 force "$\underset{\sim}{\mu}(s \upharpoonright \eta^\frown 0) = \underset{\sim}{\mu}(s_1 \upharpoonright \eta^\frown 0)$ and $\underset{\sim}{\mu}(s \upharpoonright \eta^\frown 1) = \underset{\sim}{\mu}(s_1 \upharpoonright \eta 1)$", so

$$t_1 \Vdash \text{“}\underset{\sim}{\mu}(s_1 \upharpoonright \eta^\frown 0) < \mu(s_1 \upharpoonright \eta^\frown 1)\text{” and } s_1 \Vdash \text{“}\underset{\sim}{\mu}(s_1 \upharpoonright \eta^\frown 0) \geqslant \underset{\sim}{\mu}(s_1 \upharpoonright \eta^\frown 1)\text{”.}$$

Choose $t_2 \leqslant t_1$ and rational numbers r_1, r_2 and ε such that $t_2 \Vdash$ "$r_1 \leqslant \underset{\sim}{\mu}(s_1 \upharpoonright \eta^\frown 0) < r_1 + \varepsilon$ and $r_2 < \underset{\sim}{\mu}(s_1 \upharpoonright \eta^\frown 1)$" and $r_1 + 2\varepsilon < r_2$. Define $s_2 = s_1 \cup \{\sigma \in t_2$: σ extends a maximal node of $s_1\}$. As above,

$$s_2 \Vdash \text{“}\underset{\sim}{\mu}(s_2 \upharpoonright \eta^\frown 0) \geqslant \underset{\sim}{\mu}(s_2 \upharpoonright n^\frown 1)\text{”}$$

and

$$t_2 \Vdash \text{"}r_1 \leqslant \underline{\mu}(s_2 \restriction \eta^\frown 0) < r_1 + \varepsilon \text{ and } r_2 < \underline{\mu}(s_2 \restriction \eta^\frown 1)\text{"}.$$

Notice that $(t_2, s_2) \in R_r$, where $r = r_1 + r_2$.

Since R_r is countable, there is an M-generic filter J on R_r in $M[G]$ which contains (r_2, s_2). Let A and B be the interpretations of $\underline{A}$ and $\underline{B}_r$ via J. Both A and B come from M-generic filters on Q by Lemma 3 of §2. Let μ^A and μ^B be the corresponding interpretations of $\underline{\mu}$. By the definition of $\underline{\mu}$, μ_A extends μ^A and μ_B extends μ^B.

$$\begin{aligned}\mu_A(B \restriction \eta^\frown 0) &\leqslant \mu_A(s \restriction \eta^\frown 0) < r_1 + \varepsilon < r_2 - \varepsilon = r - (r_1 + \varepsilon)\\ &= \mu_A(B) - (r_1 + \varepsilon) < \mu_A(B) - \mu_A(B \restriction \eta^\frown 0) = \mu_A(B \restriction \eta^\frown 1).\end{aligned}$$

On the other hand,

$$\begin{aligned}\mu_A(B \restriction \eta^\frown 0) &= \mu_A(B)\mu_B(B \restriction \eta^\frown 0) = \mu_A(B)\mu_B(s_2 \restriction \eta^\frown 0)\\ &\geqslant \mu_A(B)\mu_B(s_2 \restriction \eta^\frown 1) = \mu_A(B)\mu_B(B \restriction \eta^\frown 1) = \mu_A(B \restriction \eta^\frown 1),\end{aligned}$$

which is a contradiction. □

To conclude the proof of the theorem, choose $t, s \in Q$ of the same height with $s \subseteq t$ such that $t \Vdash$ "$\underline{\mu}(s \restriction \eta^\frown 0)$ and $\underline{\mu}(s \restriction \eta^\frown 1)$ are positive", where η is the stem of s. This is possible since $M^Q \Vdash$ "$\underline{\mu}$ vanishes on points". Now choose $t_1 \leqslant t$ and $s_1 \leqslant s$ of the same height such that $s_1 \subseteq t_1$ and $t_1 \Vdash$ "$0 < \underline{\mu}(s_1 \restriction \eta^\frown 0) < \underline{\mu}(s_1 \restriction \eta^\frown 1)$". By Claim 4,

$$s_1 \Vdash \text{"}\underline{\mu}(\eta^\frown 0) = \underline{\mu}(s_1 \restriction \eta^\frown 0) < \underline{\mu}(s_1 \restriction \eta^\frown 1) = \underline{\mu}(\eta^\frown 1)\text{"}.$$

Finally, choose $t_2 \leqslant t_1$ and $s_2 \leqslant s_1$ of the same height such that $s_2 \subseteq t_2$ and $t_2 \Vdash$ "$\underline{\mu}(s_2 \restriction \eta^\frown 0) > \underline{\mu}(s_2 \restriction \eta^\frown 1)$". Using Claim 4 again with 0 and 1 reversed,

$$s_2 \Vdash \text{"}\underline{\mu}(\eta^\frown 0) = \underline{\mu}(s_2 \restriction \eta^\frown 0) > \underline{\mu}(s_2 \restriction \eta^\frown 1) = \underline{\mu}(\eta^\frown 1)\text{"},$$

which is a contradiction, since $s_2 \leqslant s_1$.

4. Ulam's problem with the Axiom of Choice. Assume μ is a Borel measure on the unit interval. I use μ^* for the outer measure defined from μ: for $X \subseteq [0,1]$, $\mu^*(X) = \inf\{\mu(Y): Y \text{ is Borel and } X \subseteq Y\}$. Recall that for any $X \subseteq [0,1]$, μ^* determines a Borel measure ν on X by $\nu(Y) = \mu^*(Y)$ for Y Borel in X. A subset X of the unit interval is of *universal measure zero* if $\mu^*(X) = 0$ for all Borel measures μ. The Hausdorff gap **[1]** is an example of an uncountable set of universal measure zero.

Assuming the Axiom of Choice, we see, by the simple proof below, that there is a maximal family $\mathscr{A}$ for which a coherent family exists. Note that if A is a subset of the unit interval which is of universal measure zero, then A cannot be in the domain of a coherent system.

THEOREM (ZFC). *There is a coherent system μ_A $(A \in \mathscr{A})$, where $\mathscr{A}$ consists of all subsets of the unit interval which are not of universal measure zero.*

PROOF. Let ν_α $(\alpha \in 2^\omega)$ list all Borel measures on the unit interval. For $A \in \mathscr{A}$ define μ_A by $\mu_A(B) = \nu^*_\alpha(B)/\nu^*_\alpha(A)$, where α is the least ordinal such that $\nu^*_\alpha(A) > 0$. □

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