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MANIFOLDS M^n OF RANK $n-1$

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ABSTRACT. The determination of which closed orientable C^2 n-manifolds have rank $n-1$ is completed.

M^n will denote a closed orientable C^2 manifold. The rank of M^n is defined to be the maximal number of independent commuting vector fields on M^n. M^n has rank $n-1$ is equivalent to the existence on M^n of a C^2 locally free $\mathbf{R}^{n-1}$ action. In Theorem 1 we determine which manifolds admit free $T^k \times \mathbf{R}^l$ actions, where $k+l=n-1$. Combining Theorem 1 with several previously known results about manifolds of rank $n-1$, we obtain

THEOREM 2. *M^n has rank $n-1$ if and only if M^n is homeomorphic to a T^{n-1} bundle over S^1.*

THEOREM 1. *Suppose $T^k \times \mathbf{R}^l$ acts freely on M^{k+l+1}. If $l \geqslant 2$, then M^{k+l+1} is homeomorphic to the torus T^{k+l+1}. If $l=1$ then M^{k+2} is a principal T^k bundle over T^2.*

PROOF OF THEOREM 2. Sacksteder [**Sa**], Chatelet and Rosenberg [**CR**], and the author [**T**] give a detailed account of prior results about rank $n-1$ manifolds. We will summarize these results very briefly. In [**Sa**] it is shown that a locally free $\mathbf{R}^{n-1}$ action without compact orbits determines a free $T^k \times \mathbf{R}^{n-k-1}$ action for some k. By Theorem 1 we get that M^n is a T^{n-1} bundle over S^1 (the principal T^k bundle is trivial over an S^1 factor of T^2). In [**CR**] it is shown that a locally free $\mathbf{R}^{n-1}$ action with a compact orbit is a T^{n-1} bundle over S^1 and any such bundle admits such an action.

PROOF OF THEOREM 1. We briefly recall part of the discussion in [**T**]. M^{k+l+1} is a principal T^k bundle over $B = M^{k+l+1}/T^k$. The $T^k \times \mathbf{R}^l$ action on M^{k+l+1} projects to a free $\mathbf{R}^l$ action on B. Results of Novikov [**N**], Rosenberg [**R**], and Joubert and Moussu [**JM**] show that B is homeomorphic to T^{l+1}. There are independent commuting vector fields $\tilde{X}_i$ on M^{k+l+1} which commute with the T^k action and project to l independent commuting vector fields X_i on B. There exists a connection on this T^k bundle for which all $\tilde{X}_i$ are horizontal. It will be seen that it suffices to prove Theorem 1 in the case $k=1$ since the T^k bundle is a fiber product of k S^1 bundles. Let Ω be the curvature form for the S^1 bundle. Then $\Omega(\tilde{X}_i, \tilde{X}_j) = 0$ for

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all $1 \leqslant i, j \leqslant l$. Ω projects to a closed 2-form α on B such that the cohomology class $[\alpha]$ is the Chern class of the S^1 bundle. Let $j\colon \mathbf{R}^l \to B$ denote the inclusion of an $\mathbf{R}^l$ orbit. Then $j^*\alpha = 0$ because $\alpha(X_i, X_j) = \Omega(\tilde{X}_i, \tilde{X}_j)$.

By Sacksteder [**Sa**] there is a transversal invariant measure for the orbit foliation of the $\mathbf{R}^l$ action on B since the simply connected leaves have no holonomy. According to Ruelle and Sullivan [**RS**, p. 320] and Sullivan [**Su**, I, §3], the invariant measure allows one to define a foliation cycle $Z\colon H^l(B; \mathbf{R}) \to \mathbf{R}$ as follows: Let η be a closed l-form on B such that $\eta = \Sigma\eta_i$, where each η_i is supported in a foliation flow box of the form $D^l \times D^1$, where $D^l \times \{y\}$ is a component of the intersection of a leaf with the flow box. One integrates η_i over each $D^l \times \{y\}$ (the foliation is oriented by the $\mathbf{R}^l$ action), and the resulting function of y is then integrated against the invariant measure on D^1. One adds the results from each flow box to obtain $Z([\eta])$. The result depends only on the oriented foliation and the invariant measure.

Let $\eta = \alpha \wedge \beta$, where β is a closed $(l-2)$-form and α denotes the Chern form as above. Then $Z([\alpha \wedge \beta]) = 0$ because $j^*(\alpha \wedge \beta) = 0$, where j is the inclusion of any leaf in B.

If the foliation is defined by a closed 1-form ω, then ω defines a transversal invariant measure and, hence, a foliation cycle Z. By Sullivan [**Su**, Theorem I.13] $[\omega] \in H^1(B; \mathbf{R})$ corresponds to Z by duality, i.e., $Z([\eta]) = \int_M \omega \wedge \eta$. By [**Sa**] there is a change of atlas on B in which the invariant measure for the orbit foliation above is given by a closed 1-form ω. The change of atlas does not change the invariant measure or the foliation cycle. Although α may not be differentiable in the new atlas, the Chern class $[\alpha]$ remains unchanged.

By the discussion in the above two paragraphs we see that $[\omega] \wedge [\alpha] \wedge [\beta] = 0$ for any β. Therefore, $[\omega] \wedge [\alpha] = 0$ in $H^3(B; \mathbf{R})$.

Since B is homemorphic to T^{l+1}, we have that $H^*(B; \mathbf{R})$ is the exterior algebra on $H^1(B; \mathbf{R})$. Let $f, e_1, \ldots, e_l$ be a basis for $H^1(B; \mathbf{Z})$. We can write $[\omega] = \lambda[f + \Sigma_1^l \lambda_t e_t]$ for some real numbers λ, λ_t. We can write

$$[\alpha] = \sum_{1 \leqslant i < j \leqslant l} n_{ij} e_i \wedge e_j + \sum_{1 \leqslant s \leqslant l} m_s e_s \wedge f,$$

where n_{ij} and m_s are integers (the Chern class is an integral class). Assuming $l \geqslant 2$, the equation $[\omega] \wedge [\alpha] = 0$ in $H^3(B; \mathbf{R})$ implies that $n_{ij} + \lambda_i m_j - \lambda_j m_i = 0$ for all $1 \leqslant i < j \leqslant l$.

Let F, E_t, $1 \leqslant t \leqslant l$, be the basis for $H_1(B; \mathbf{Z})$ dual to f, e_t. Let $c_{ij} = n_{ij}F + m_j E_i - m_i E_j$. Then $\int_{c_{ij}} \omega = 0$. By Rosenberg [**R**, Theorem 5] since the leaves are simply connected, either $c_{ij} = 0$ or c_{ij} is homotopic to a loop which is transverse to the orbits. Since $\int_{c_{ij}} \omega = 0$, the loop cannot be transverse to the orbits. Hence $c_{ij} = 0$, which means $n_{ij} = 0$, $m_i = 0$, $m_j = 0$ for all $1 \leqslant i < j \leqslant l$. Therefore, $[\alpha] = 0$ and the S^1 bundle is trivial. Therefore, the original T^k bundle is trivial and M^{k+l+1} is homeomorphic to $T^k \times T^{l+1}$.

Remark. The above result of Rosenberg corresponds to the fact that the periods of the 1-form ω are independent when $\mathbf{R}$ is viewed as a vector space over the rationals. That is why there are no nonzero integral classes in the kernel of $c \to \int_c \omega$.

When $l = 1$, $B = T^2$ and M^{k+2} is a principal T^k bundle over T^2. These manifolds admit a free $T^k \times \mathbf{R}$ action (see [T]). This completes the proof of Theorem 1.

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