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SOLVABILITY OF DIFFERENTIAL EQUATIONS WITH LINEAR COEFFICIENTS OF NILPOTENT TYPE

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ABSTRACT. Let L be the vector field on $\mathbf{R}^n$ associated with a real nilpotent $(n \times n)$-matrix. It is shown that L regarded as a differential operator defines a surjective mapping of the space $\mathscr{S}'$ of tempered distributions onto itself; i.e. $L\mathscr{S}'(\mathbf{R}^n) = \mathscr{S}'(\mathbf{R}^n)$. Replacing $\mathscr{S}'$ by the space $\mathscr{D}'$ of ordinary distributions, this is not true in general.

1. Introduction. Let L be the infinitesimal transformation on $\mathbf{R}^n$ associated with a real $(n \times n)$-matrix X; i.e. L is given by

$$L\varphi(x) = \frac{d}{dt}\varphi(e^{-tX}x)|_{t=0} = -\sum_{i=1}^{n}(Xx)_i\frac{\partial\varphi}{\partial x_i}(x),$$

$\varphi \in C^\infty(\mathbf{R}^n)$, $x \in \mathbf{R}^n$, $(Xx)_i$ the ith component of Xx. Let us regard L as a linear mapping of $\mathscr{D}$ into itself, where $\mathscr{D}$ is the space of all C^∞-functions with compact support on $\mathbf{R}^n$. Furthermore, we also regard L as a linear mapping of $\mathscr{D}'$ into itself defined in the usual way by continuous extension, where $\mathscr{D}'$ is the dual space of $\mathscr{D}$, which is the space of (ordinary) distributions. The distributions annihilating the image $L\mathscr{D}$ of L are just the distributions invariant under e^{tX}, $t \in \mathbf{R}$. We write $\mathscr{D}'_X = (L\mathscr{D})^\perp$.

We ask the following questions closed related with each other:

(i) Is $L\mathscr{D}$ closed in $\mathscr{D}$?

(ii) How do we characterize the invariant distributions? Is there a canonical fundamental set?

(iii) Is the differential operator L solvable in some sense? That means: When has the equation $Lu = f$ a solution u?

Essentially this is a special case of the problem investigated in **[9]**. Nevertheless, it seems to be very difficult to answer these questions in general. (See the examples below.)

In **[9]**, questions (i) and (ii) are studied in a more general framework: Let M be a differentiable manifold and $\mathscr{L}$ a Lie algebra of infinitesimal transformations on X. Then the set $\mathrm{Div}(\mathscr{L})$ of all finite sums $\sum L_i\varphi_i$, where $L_i \in \mathscr{L}$ and $\varphi_i \in \mathscr{D}(M)$, is characterized in some special situations. Particularly, if at each point of M the vector

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fields in $\mathscr{L}$ give an m-dimensional subspace of the tangent space to M, it is claimed in [**9**, Theorem 2], that (under some additional assumptions)

(a) $\mathrm{Div}(\mathscr{L})$ is closed;

(b) $g \in \mathscr{D}(M)$ belongs to $\mathrm{Div}(\mathscr{L})$ iff g is annihilated by the invariant measures on the integral manifolds.

In [**5**, §4], a counterexample to (b) is given. (See also [**1**, §2.1] for new versions of [**9**, Theorem 2]; see [**2–4**].) In Example 2 we shall see that (a) is false, too.

Coming back to our special situation, we deal with the case that X is a nilpotent matrix. In view of Example 2 we are suggested to work with $\mathscr{S}$ and $\mathscr{S}'$ rather than with $\mathscr{D}$ and $\mathscr{D}'$, where $\mathscr{S}$ is the Schwartz space of rapidly decreasing smooth functions and $\mathscr{S}'$ its dual space, which is the space of tempered distributions. Working with $\mathscr{S}$ and $\mathscr{S}'$ we get satisfactory answers to our questions.

Let us state the results: We call a pair (v, w) of vectors in $\mathbf{R}^n$ X-admissible if $v \neq 0$ and $Xw \neq 0$. For X-admissible pairs (v, w) and for integer $k \geqslant 0$ we define the tempered invariant distribution

$$T_{v,w}^{(k)}(\varphi) := \int_{\mathbf{R}} \nabla_v^k(\varphi \circ e^{tX})(w)\, dt, \qquad \varphi \in \mathscr{S},$$

where ∇_v denotes the directional derivative. We write $\mathscr{M}_X$ (resp. $\mathscr{M}_X^\circ$) for the set of all $T_{v,w}^{(k)}$ (resp. $T_{v,w}^{(\circ)}$). Clearly, the elements of $\mathscr{M}_X^\circ$ are just the invariant measures on the nontrivial $e^{\mathbf{R}X}$-orbits in $\mathbf{R}^n$.

THEOREM. *Let X be an arbitrary nonzero real nilpotent $(n \times n)$-matrix and let L be the infinitesimal transformation associated with X.*

Then $L\mathscr{S}$ is closed in $\mathscr{S}$. Moreover the invariant tempered distributions can be characterized as follows:

For $\mathrm{rank}(X) = 1$ *the invariant orbital measures form a fundamental set. For* $\mathrm{rank}(X) > 1$ *the set $\mathscr{M}_X$ is fundamental.* (*Note that in general the invariant orbital measures do not form a fundamental set according to* [**5**, §1].)

COROLLARY. *The differential operator L regarded as a mapping of $\mathscr{S}'$ into itself is surjective.*

To prove the theorem, Lemma 2.2 of [**6**] is crucially used. For the convenience of the reader it is cited here:

LEMMA A. *Suppose that $\mathbf{R}^{n-1} \triangleq \{x \in \mathbf{R}^n | x_1 = 0\}$ is X-invariant and contains the kernel of X. Let L' be the infinitesimal transformation on $\mathbf{R}^{n-1}$ associated with the restriction of X to $\mathbf{R}^{n-1}$. Let $\mathscr{M} \subseteq \mathscr{S}_X'(\mathbf{R}^n)$ be a set of invariant tempered distributions containing the invariant measures on the orbits in $\{x_1 \neq 0\}$ and satisfying the following conditions*:

(i) *if $\varphi \in \mathscr{S}(\mathbf{R}^n)$ and $x_1\varphi \in \mathscr{M}^\perp$, then $\varphi \in \mathscr{M}^\perp$*;

(ii) *if $\varphi \in \mathscr{M}^\perp$, then the restriction of φ to $\mathbf{R}^{n-1}$ belongs to $L'\mathscr{S}(\mathbf{R}^{n-1})$.*

Then $\mathscr{M}^\perp = L\mathscr{S}(\mathbf{R}^n)$.

2. Examples. To explain the area of validity of the assertions in the theorem we give two examples. Example 1 shows that the assertions do not need to be valid if X is not nilpotent. In Example 2 we see that $\mathscr{S}$ cannot be replaced by $\mathscr{D}$ or $\mathscr{E}$.

EXAMPLE 1. (Compare to Example 1 of [**9**].) Let α, β be real numbers with α/β irrational and let

$$X = \begin{pmatrix} 0 & -\alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta \\ 0 & 0 & \beta & 0 \end{pmatrix}.$$

Then the one-parameter-subgroup e^{tX}, $t \in \mathbf{R}$, in SO(4) is not closed. By [**8**, Chapter IV, Theorem D], it is easily seen that the closure $\overline{L\mathscr{D}}$ of $L\mathscr{D}$ in $\mathscr{D}$ is just the set of all test functions g for which $\int_H g(bx)\,db = 0$ for all $x \in \mathbf{R}^4$, where H is the closure of $e^{\mathbf{R}X}$ in SO(4). Using a rotation invariant partition of unity we conclude that the closure $\overline{L\mathscr{S}}$ of $L\mathscr{S}$ in $\mathscr{S}$ is also the set of all $g \in \mathscr{S}$ for which $\int_H g(bx)\,db = 0$.

For $x \in \mathbf{R}^4$ we write $x = (y, z)$ where $y, z \in \mathbf{R}^2$. By polar decomposition we have $y = r \cdot e(\sigma)$ where $e(\sigma) := (\cos\sigma, \sin\sigma)$; similarly $z = s \cdot e(\tau)$. Let γ be a test function on $]0, \infty[$ with $\gamma(1) = 1$. If $h(y, z)$ is a C^∞-function on $\{x \in \mathbf{R}^4 \mid |y| = |z| = 1\}$ for which $\int_0^{2\pi}\int_0^{2\pi} h(e(\sigma), e(\tau))\,d\sigma\,d\tau = 0$, then the function $g(x) := \gamma(r)\gamma(s)h(e(\sigma), e(\tau))$ belongs to $\mathscr{D}$ and satisfies the condition $\int_H g(bx)\,db = 0$ for all $x \in \mathbf{R}^4$. Assuming $g \in L\mathscr{D}$ or $g \in L\mathscr{S}$, say $g = L\varphi$, we receive

$$h(e(\sigma), e(\tau)) = \alpha\frac{\partial}{\partial\sigma}\Psi(e(\sigma), e(\tau)) + \beta\frac{\partial}{\partial\tau}\Psi(e(\sigma), e(\tau))$$

for the C^∞-function $\Psi(y, z) := -\varphi(y, z)$ on $\{|y| = |z| = 1\}$. But this is not possible for every h, when α/β is a Liouville number [**9**, Example 1]. Thus neither $L\mathscr{D}$ nor $L\mathscr{S}$ is closed in $\mathscr{D}$ and $\mathscr{S}$, respectively, whenever α/β is a Liouville number.

EXAMPLE 2. Let

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

It is shown in [**5**] that there are invariant tempered distributions on $\mathbf{R}^3$, which cannot be approximated by linear combinations of invariant orbital measures. More precisely, let $\alpha, \beta, \gamma \in \mathscr{D}(\mathbf{R})$ such that $\alpha(0) = 0$ and $\beta(-t) = -\beta(t)$ for $t \in \mathbf{R}$ and put $g(x) := \alpha(x_1)\beta(x_2)\gamma(x_3) \in \mathscr{D}(\mathbf{R}^3)$. Then g is annihilated by all invariant orbital measures, but if $k > 0$ we have $T_{v,w}^{(k)}(g) \neq 0$ for $\alpha = x_1^k\eta$, $\eta \in \mathscr{D}(\mathbf{R})$, such that $\eta(0) \neq 0$, and for suitable functions β, γ, where $v = (1,0,0)$ and $w = (0,1,0)$. Therefore, assertion (b) of Theorem 2 in [**9**] (see introduction) cannot be valid. However, at this point it is not yet clear if $L\mathscr{D}$ is closed or not.

To answer this question we select α, β, γ such that $\beta \geqslant 0$ on $]0, \infty[$, $\beta(1) > 0$, $\gamma \geqslant 0$, $\gamma(0) > 0$, $\alpha > 0$ on the open interval $]0, 1[$ and $\alpha = 0$ outside of $]0, 1[$. Now we find a sequence (α_ν) converging to α in $\mathscr{D}(\mathbf{R})$ with $\alpha_\nu = 0$ outside of $]\varepsilon_\nu, 1[$, $\varepsilon_\nu > 0$. Then the sequence $g_\nu(x) := \alpha_\nu(x_1)\beta(x_2)\gamma(x_3)$ converges to $g(x) := \alpha(x_1)\beta(x_2)\gamma(x_3)$ in $\mathscr{D}(\mathbf{R}^3)$. At first we show that $g_\nu \in L\mathscr{D}$ for all ν. By [**6**, Lemma 2.6], g_ν belongs to $L\mathscr{S}$. The function $\varphi_\nu \in \mathscr{S}$ for which $g_\nu = L\varphi_\nu$ is obtained by the formula

$$\varphi_\nu(x) = \int_0^\infty g_\nu\left(x_1, tx_1 + x_2, \frac{t^2}{2}x_1 + tx_2 + x_3\right)dt.$$

(See [**6**, §2.1].) Using this formula, the fact that φ_ν has compact support is verified by a routine calculation keeping in mind that $g_\nu(x) = 0$ for $x_1 < \varepsilon_\nu$. From $g_\nu \in L\mathscr{D}$ we receive $g \in \overline{L\mathscr{D}}$. Assuming $g \in L\mathscr{D}$, say $g = L\varphi$, our formula implies

$$\varphi\left(\varepsilon, 0, -\frac{1}{2\varepsilon}\right) = \alpha(\varepsilon)\int_0^\infty \beta(t\varepsilon)\gamma\left(\frac{t^2}{2}\varepsilon - \frac{1}{2\varepsilon}\right)dt > 0$$

for all $0 < \varepsilon < 1$ because the integrand is $\geqslant 0$ and > 0 for $t = 1/\varepsilon$. But this is not possible, since φ has compact support. Thus $L\mathscr{D}$ is not closed in $\mathscr{D}$.

Choosing β in such a way that $\beta = 0$ on a neighbourhood of 0 we get a counterexample to assertion (a) of Theorem 2 in [**9**] (see introduction).

Moreover, it follows by functional analysis that the equation $Lu = f$ does not have a solution $u \in \mathscr{D}'$ for every $f \in \mathscr{D}'$.

Furthermore, we can conclude that $L\mathscr{E}$ is not closed in $\mathscr{E}$, where $\mathscr{E}$ is the space of all infinitely differentiable functions provided with the usual topology. Suppose that $L\mathscr{E}$ is closed in $\mathscr{E}$. Then the mapping $L\colon \mathscr{E}' \to \mathscr{E}'$ has a closed range. Therefore we can find a distribution u with compact support satisfying $Lu = g$. Applying once more [**6**, Lemma 2.6], we receive $L\varphi = g$ for the function $\varphi \in \mathscr{S}$ given by the previous formula. It follows that $L(\varphi - u) = 0$, therefore the distribution $\varphi - u$ is invariant. We determine $r \geqslant 2$ such that the support of u is contained in $\{|x_i| \leqslant r, i = 1, 2, 3\}$. Now let $\varepsilon < 1/2r$ and select a test function $\Psi \geqslant 0$ satisfying $\Psi(\varepsilon, 0, -1/2\varepsilon) > 0$ such that the support of Ψ is contained in a δ-neighbourhood of $(\varepsilon, 0, -1/2\varepsilon)$, where δ is a sufficiently small positive number, $\delta < \varepsilon/2$. For $t > 2(r+1)/\varepsilon$ the support of $\Psi \circ e^{-tX}$ is contained in $\{x_1 > \varepsilon/2, x_2 > r\}$, therefore we have

$$\langle u, \Psi\rangle = \langle u, \Psi \circ e^{-tX}\rangle = 0 \quad \text{and} \quad \langle \varphi, \Psi \circ e^{-tX}\rangle = 0$$

but $\langle \varphi, \Psi\rangle \neq 0$. Using the invariance of $\varphi - u$ we get the following contradiction:

$$0 \neq \langle \varphi, \Psi\rangle = \langle \varphi - u, \Psi\rangle = \langle \varphi - u, \Psi \circ e^{-tX}\rangle = 0.$$

3. Proofs. Let X be an arbitrary nilpotent $(n \times n)$-matrix, $X \neq 0$. After change of basis we may assume that X has the form

$$\begin{pmatrix} 0 & & & & \\ 1 & \ddots & & & \\ & \varepsilon_2 & \ddots & & \\ & & \ddots & \ddots & \\ & & & \varepsilon_{n-1} & 0 \end{pmatrix}$$

where $\varepsilon_j \in \{0, 1\}, j = 2, \ldots, n-1$.

LEMMA 1. *Let $\varphi \in \mathscr{S}$. Assume that $x_1\varphi$ is annihilated by all $T \in \mathscr{M}_X$. (We write $x_1\varphi \in \mathscr{M}_X^\perp$.) Then $T_{v,w}^{(k)}(\varphi) = 0$ whenever $v_1 \neq 0$ (v_1 = the first component of v).*

PROOF. By [**6**, §2.4], we have the formula $x_1 T_{v,w}^{(k)} = kv_1 T_{v,w}^{(k-1)} + w_1 T_{v,w}^{(k)}$, $k > 0$. Therefore $kv_1 T_{v,w}^{(k-1)}(\varphi) + w_1 T_{v,w}^{(k)}(\varphi) = 0$. For $w_1 = 0$ the assertion follows immediately. For $w_1 \neq 0$ we conclude $T_{v,w}^{(0)}(\varphi) = 0$ from $0 = \langle T_{v,w}^{(0)}, x_1\varphi\rangle = w_1\langle T_{v,w}^{(0)}, \varphi\rangle$ and proceed by induction on k.

LEMMA 2. *Let $\varphi \in \mathscr{S}$. If $x_1\varphi \in \mathscr{M}_X^\perp$ then $\varphi \in \mathscr{M}_X^\perp$.*

PROOF. By Lemma 1, we have only to prove that $T_{v,w}^{(k)}(\varphi) = 0$ whenever $v_1 = 0$. Now, by Lemma 1,

$$0 = T_{v^{(\nu)},w}^{(k)}(\varphi) = \int \nabla_{v^{(\nu)}}^k(\varphi \circ e^{tX})(w)\, dt$$

where $v^{(\nu)} = v + (1/\nu, 0, \ldots, 0)$, $\nu \in \mathbf{N}$. Using the formula

$$\nabla_{v^{(\nu)}}^k = \sum_{m=0}^{k} \binom{k}{m} \frac{1}{\nu^m} \frac{\partial^m}{\partial x_1^m} \nabla_v^{k-m}$$

we conclude

$$0 = \sum_{m=0}^{k} \binom{k}{m} \frac{1}{\nu^m} \int \frac{\partial^m}{\partial x_1^m} \nabla_v^{k-m}(\varphi \circ e^{tX})(w)\, dt.$$

For $\nu \to \infty$ we get

$$0 = \int \nabla_v^k(\varphi \circ e^{tX})(w)\, dt = T_{v,w}^{(k)}(\varphi).$$

LEMMA 3. *Let the rank of X be equal to 1. Let $\varphi \in \mathscr{S}$. Suppose that $\varphi \in (\mathscr{M}_X^\circ)^\perp$ and $\varphi(x) = 0$ whenever $x_1 = 0$. Then $\varphi \in L\mathscr{S}$.*

PROOF. Compare to [**6**, Lemma 2.5]. We get $\varphi = x_1\chi$ where $\chi \in (\mathscr{M}_X^\circ)^\perp$. Therefore $\int_{\mathbf{R}} \chi(x)\, dx_2 = 0$. Thus there is a function $\psi \in \mathscr{S}$ such that $\chi = \partial\psi/\partial x_2$. We get $\varphi = x_1 \partial\psi/\partial x_2 = -L\psi$.

LEMMA 4. *Let the rank of X be equal to 2 and let L' be the infinitesimal transformation on $\mathbf{R}^{n-1} \stackrel{\wedge}{=} \{x \in \mathbf{R}^n | x_1 = 0\}$ associated with the matrix*

$$\begin{pmatrix} 0 & & & \\ \varepsilon_2 & \ddots & & \\ & \ddots & \ddots & \\ & & \varepsilon_{n-1} & 0 \end{pmatrix}.$$

Then for every $\varphi \in (\mathscr{M}_X^\circ)^\perp$ the restriction φ' to $\mathbf{R}^{n-1}$ belongs to $L'\mathscr{S}(\mathbf{R}^{n-1})$.

PROOF. Obviously it is sufficient to consider the two cases $\varepsilon_j = \delta_{2,j}$ and $\varepsilon_j = \delta_{3,j}$, $j = 2, \ldots, n-1$.

Let $\varepsilon_j = \delta_{2,j}$. By Lemma 3, we have only to prove that $\varphi'(x_2, x_3, \ldots, x_n) = 0$ whenever $x_2 = 0$. By assumption, for all $\nu \in \mathbf{N}$,

$$0 = \int \varphi\left(e^{tX}\left(\frac{1}{\nu^2}, 0, x_3, \ldots, x_n\right)\right) dt = \nu \int \varphi\left(\frac{1}{\nu^2}, \frac{t}{\nu}, \frac{t^2}{2} + x_3, x_4, \ldots, x_n\right) dt.$$

For $\nu \to \infty$ we get $\int \varphi(0, 0, t^2/2 + x_3, \ldots, x_n)\, dt = 0$ for all $x_3, \ldots, x_n$. It is proved in [**6**, Lemma 2.6] that from this it follows that $\varphi(0, 0, x_3, \ldots, x_n) = 0$ for all $x_3, \ldots, x_n$.

Let $\varepsilon_j = \delta_{3,j}$. By Lemma 3, we have only to prove that $\varphi'(x_2, x_3, \ldots, x_n) = 0$ whenever $x_3 = 0$. By assumption, for all $\nu \in \mathbf{N}$ and for every nonzero vector $z = (z_1, z_2) \in \mathbf{R}^2$ we have

$$0 = \int \varphi\left(e^{tX}\left(\frac{z_1}{\nu}, x_2, \frac{z_2}{\nu}, x_4, \ldots, x_n\right)\right) dt$$

$$= \nu \int \varphi\left(\frac{z_1}{\nu}, x_2 + tz_1, \frac{z_2}{\nu}, x_4 + tz_2, \ldots, x_n\right) dt.$$

For $\nu \to \infty$ we get $\int \varphi(0, x_2 + tz_2, 0, x_4 + tz_2, \ldots, x_n)\, dt = 0$ for all $x_2, x_4, \ldots, x_n$. This means that the one-dimensional Radon transform of the function $(x_2, x_4) \to \varphi(0, x_2, 0, x_4, \ldots, x_n)$ is identically 0 for all $x_5, \ldots, x_n$. Now the assertion follows. (See [7, Chapter I, §6].)

PROOF OF THE THEOREM. For rank$(X) = 1$ the Theorem is just Lemma 3. For rank$(X) > 1$ we prove $\mathscr{M}_X^{\perp} = L\mathscr{S}$ by induction on rank(X). For rank$(X) = 2$ the assertion follows from Lemma A, using Lemmas 2 and 4. For rank$(X) > 2$ the assertion follows from Lemma A, using Lemma 2 and the induction hypothesis.

PROOF OF THE COROLLARY. By the Theorem, we only have to prove that $L: \mathscr{S} \to \mathscr{S}$ is injective. Now, if $L\varphi = 0$ for $\varphi \in \mathscr{S}$, then φ must be invariant because of $\mathscr{S}_X' = (L\mathscr{S})^{\perp}$; i.e. φ is constant on the orbits. In view of the fact that "almost all" orbits are unbounded this is not possible except for $\varphi = 0$.

REFERENCES

1. R. Barrà, *Divergences et distributions invariantes*, Thèse de 3^e cycle, n^o 723, Université de Poitiers, 1978.

2. ______, *Fonctions divergences et distributions invariantes*, Bull. Sci. Math. (2) **105** (1981), 49–71.

3. ______, *Fonctions divergences et distributions invariantes*. II, Bull. Sci. Math. (2) **107** (1983), 209–217.

4. ______, *Distributions invariantes*. III, Preprint.

5. J. Dixmier, *Sur les distributions invariantes par un groupe nilpotent*, C. R. Acad. Sci. Paris Ser. A **285** (1977), 7–10.

6. R. Felix, *Das Syntheseproblem für invariante Distributionen*, Invent. Math. **65** (1981), 85–96.

7. S. Helgason, *The Radon transform*, Progress Math. 5, Birkhäuser, Boston-Basel-Stuttgart, 1980.

8. C. S. Herz, *Analyse harmonique à plusieurs variables*, Publ. Math. Orsay, 1965.

9. ______, *Functions which are divergences*, Amer. J. Math. **92** (1970), 641–656.

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