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ON PROPERLY EMBEDDING PLANES IN ARBITRARY 3-MANIFOLDS

E. M. BROWN AND C. D. FEUSTEL

ABSTRACT. We prove an analog of the loop theorem for an arbitrary noncompact 3-manifold. In particular, we show that the existence of a "nontrivial" proper map of a plane into a 3-manifold implies the existence of a nearby nontrivial embedding of a plane into the 3-manifold.

Introduction. In [2] we showed that if M is an eventually end-irreducible 3-manifold and $f\colon \mathbf{R}^2 \to M$ is a proper essential map, then there is a proper essential embedding $g\colon \mathbf{R}^2 \to M$. In this paper we remove the restriction to manifolds which are eventually end-irreducible. This permits us to choose the image of g to lie in a preassigned neighborhood of the image of f. This could not be done with the weaker theorem, for even when M is eventually end-irreducible, a regular neighborhood of $f(\mathbf{R}^2)$ need not be. We also extend the theorem in another direction; roughly, we show that if a normal subgroup of the fundamental group of the complement of a compact set in M is given and large loops around the origin in $\mathbf{R}^2$ are mapped by f to homotopy classes not in the subgroup, then g can be chosen to have the same property.

The proof of the main theorem, (2.2), parallels that of [2]. At one point, [2] makes major use of the eventually end-irreducible hypothesis. In this paper we prove Lemma (2.1) to substitute where eventually end-irreducible was used before. We believe the technique used in Lemma (2.1) is new. We expect it to be generally applicable to 3-manifolds which are not eventually end-irreducible.

1. Notational conventions and a preliminary lemma. We work in the category of simplicial complexes and piecewise linear maps. A map $f\colon X \to Y$ is proper if $f^{-1}(C)$ is compact for every compact $C \subset Y$. If $X \subset Y$ we use $\mathrm{Fr}(X)$ to mean the frontier of X in Y and $\mathrm{Cl}(X)$ to mean the closure of X in Y. If M is a manifold, we use ∂M for the boundary of M. We follow Waldhausen's convention [6] on regular neighborhoods; specifically, if $X \subset Y$, choose a triangulation of Y in which all previously mentioned subspaces are subcomplexes, and let $U(X)$ be the simplicial neighborhood of X in the second barycentric subdivision.

If M is a 3-manifold, an exhausting sequence for M will be a sequence $\{M_n\}$ of compact 3-manifolds in M with $M_n \subset M_{n+1} - \mathrm{Fr}(M_{n+1})$, $M_n \cap \partial M$ a 2-manifold,

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and $\bigcup_n M_n = M$. The word "proper" is used in two senses; here, where applicable, we mean both. In particular, a 2-manifold F is properly embedded in a 3-manifold M if the inclusion map $F \to M$ is proper and $F \cap \partial M = \partial F$. If F is a 2-manifold properly embedded in a 3-manifold M, Jaco denotes the result of cutting M over F, i.e. $\mathrm{Cl}(M - U(F))$, $\alpha_F(M)$. We use $\alpha(F, M)$ for this to avoid double subscripts. If M, N $(M \subset N)$ are 3-manifolds and $F \subset N$ is a 2-manifold with $F \cap \mathrm{Fr}(M) = \partial F$, we also write $\alpha(F, M)$ for $M \cup U(F)$ (we still refer to this as cutting M over F). We also write $\alpha(F, M)$, where F is a finite family of 2-manifolds and we are cutting over them in sequence. When the order in which we cut matters, we give F as an ordered family.

We shall write our fundamental groups without basepoint, as our use of fundamental groups will be independent of the choice of basepoint. If $f\colon A \to M$ is a map of a 1-sphere into M, we use $[f]$ for the homotopy class in $\pi_1(M)$ of f with some choice of basepoint and some orientation of S^1. Notice that if G is a normal subgroup of $\pi_1(M)$ then $[f] \notin G$ makes sense. An unlabeled map between fundamental groups will always be induced by inclusion.

We use the loop theorem [**5**] without further reference.

LEMMA (1.1) *Let M be a 3-manifold and let M_0 be a compact submanifold. There is an exhausting sequence $\{M_m\}$ for M, with $M_0 \subset M_m$, and for each m there is a finite family $\mathscr{D}_m$ of disks properly embedded in $M_m - M_0 - \partial M$ so that* $\mathrm{Fr}(\alpha(\mathscr{D}_m, M_m))$ *is incompressible in $M - M_0$.*

We do not assert that $\{\alpha(\mathscr{D}_m, M_m)\}$ exhausts M; that is, generally, impossible. The point is that all the members of $\mathscr{D}_m$ are contained in M_m.

PROOF. Let $\{M'_m\}$ be an exhausting sequence for M. Suppose, by induction, that for all $1 \leqslant m < n$ we have chosen M_m and $\mathscr{D}_m$ as above and, in addition, $M'_m \subset M_m$ (this last is so that the sequence we end up with exhausts M). If M_n is any compact manifold with $M_{n-1} \cup M'_n \subset M_n - \mathrm{Fr}(M_n)$, it is a standard argument that one can construct a finite sequence of disks $\mathscr{D}_n = (D_1, \ldots, D_k)$ in $M - M_0$ so that $\mathrm{Fr}(\alpha(\mathscr{D}_n, M_n))$ is incompressible in $M - M_0$. We may not, however, be able to choose all the disks in M_n. We show that if M_n is chosen appropriately, this can be done; choose M_n and $\mathscr{D}_n$ so that $\mathscr{D}_n$ has the least number of disks possible not contained in M_n. Let D_i be the first in the sequence so that D_i is not contained in M_n. We may assume that $D_1, \ldots, D_{i-1}$ are disjoint and for $j < i$ that D_i meets $U(D_j)$ in disks parallel to D_j, i.e. in disks whose boundaries are essential in the annulus $U(D_j) \cap \mathrm{Fr}(M_n)$. Let $M''_n = M_n \cup U(D_i)$. Then $\alpha((D_1, \ldots, D_i), M_n)$ can be constructed from M''_n by cuts parallel to the disks $D_1, \ldots, D_{i-1}$. It follows that all of $D_1, \ldots, D_k$ are contained in M_n and the lemma is proved.

If D is a properly embedded disk in a 3-manifold M there is a homeomorphism of $U(D)$ onto $D \times [-1, 1]$ carrying $U(D) \cap \partial M$ onto $\partial D \times [-1, 1]$. We identify via such a homeomorphism and use $U(D)$ and $D \times [-1, 1]$ interchangeably. In Lemma (1.1) we may assume the disks chosen so that if $D \in \mathscr{D}_m$ and $E \in \mathscr{D}_n$, $m < n$, then $U(E) \cap M_m \subset U(\bigcup \mathscr{D}_m)$ and $U(E) \cap U(D) = D \times F$, where $F \subset (-1, 1)$ is a finite set of closed intervals. We may further assume that if $p \in E \cap U(D)$, then the product line $\{p\} \times [-1,1]$ of $U(E)$ is contained in the product line $\{p\} \times [-1,1]$ of $U(D)$. We make these assumptions when we use Lemma (1.1).

2. The main lemma and the theorem. If C is a compact subset of M, by a *complementary domain* of C we mean the closure of a component of $M - C$.

LEMMA (2.1). *Let M be a 3-manifold and let f: $\mathbf{R}^2 \to M$ be a proper map. Let $C \subset M$ be compact, let A be a complementary domain of C, and let G be a normal subgroup of $\pi_1(A)$. Suppose $D_0 \subset \mathbf{R}^2$ is a disk, $f(\mathbf{R}^2 - D_0) \subset A$, and $[f|\partial D_0] \notin G$. Then there exists an exhausting sequence $\{M'_m\}$ for M, an exhausting sequence $\{D_m\}$ of disks in $\mathbf{R}^2$, and a proper map g: $\mathbf{R}^2 \to M$ so that $g(\mathbf{R}^2 - D_0) \subset A$, $[g|\partial D_0] \notin G$, and D_m is a component of $g^{-1}(M'_m)$.*

PROOF. We may assume that $f(\mathbf{R}^2) \cap \partial M$ is empty. It is sufficient to prove that if $L \subset M$ is any compact submanifold containing C then there exists a proper map g: $\mathbf{R}^2 \to M$ and a compact submanifold N of M containing L so that $f^{-1}(L) = g^{-1}(L)$, $f|f^{-1}(L) = g|f^{-1}(L)$, $g^{-1}(L)$ is contained in a component of $g^{-1}(N)$, and that component is a disk.

Suppose L is given; choose a compact submanifold M_0 of M so that:

(1) $f^{-1}(L)$ is contained in a single component of $f^{-1}(M_0)$.

Subject to the restrictions on g above, assume f has been altered so that $f^{-1}(M_0)$ has the least number of components possible. Now choose $\{M_m\}$ and $\{\mathscr{D}_m\}$ as in Lemma (1.1). Altering f outside $f^{-1}(M_0)$ by a proper homotopy and choosing a subsequence of $\{M_m\}$, we may further assume:

(2) f is in general position with respect to M_m and $\mathscr{D}_m$ for all m.

(3) $f(\mathbf{R}^2) \cap (D \times [-1,1]) = (f(\mathbf{R}^2) \cap D) \times [-1,1]$ for all $D \in \mathscr{D}_m$.

(4) $f^{-1}(M_m)$ is contained in a disk in $f^{-1}(M_{m+1})$ for $m = 0, 1, \ldots$.

Let $N_m = \alpha(\mathscr{D}_m, M_m)$. Let $A_0 = B_0$ denote the component of $f^{-1}(M_0)$ which contains $f^{-1}(L)$. Let A_m (respectively, B_m) denote the component of $f^{-1}(M_m)$ (respectively, $f^{-1}(N_m)$) which contains A_{m-1} (respectively, B_{m-1}). It follows from (2) and (3) that A_m and B_m are disks-with-holes, and from (4) that $f^{-1}(M_{m-1}) \subset A_m$, for $m = 1, 2, \ldots$. A component of ∂A_m (respectively, ∂B_m) is called an *inner* boundary component if the disk it bounds in $\mathbf{R}^2$ does not contain A_m (respectively, B_m).

We show that f can be altered so that all inner boundary components of some B_n disappear. Then B_n is a disk and contains $f^{-1}(L)$, so $N = N_n$ will prove the lemma. Let λ be an inner boundary component of B_m. If $f(\lambda)$ is an inessential loop in $\mathrm{Fr}(N_m)$, then we can alter f to carry a small neighborhood of the disk bounded by λ into a collar neighborhood of $\mathrm{Fr}(N_m)$ (which eliminates λ from the list of inner boundary components of B_m). This will certainly be the case if the disk bounded by λ contains no component of $f^{-1}(M_0)$. We show that this last is true for m sufficiently large; indeed, choose n so that B_n contains as many components of $f^{-1}(M_0)$ as possible.

By (2) and (3), the inner boundary components of B_n are of three kinds:

(5) components which are also inner boundary components of A_n,

(6) components which miss ∂A_n,

(7) components made up of arcs which f maps properly into $\bigcup \mathcal{D}_n \times \{-1,1\}$ alternating with arcs of ∂A_n.

Let λ be an inner boundary component of B_n. We show that the disk E bounded by λ contains no points of $f^{-1}(M_0)$. Since $f^{-1}(M_0)$ is contained in A_n, this is obvious if λ is of type (5). If λ is of type (6) then $f(\lambda) \subset D \times \{-1,1\}$ for some $D \in \mathcal{D}_n$: since $f^{-1}(M_0)$ has the least number of components possible, it again follows that E misses $f^{-1}(M_0)$. Suppose λ is of type (7) and E contains a component K of $f^{-1}(M_0)$. By our choice of n, $K \not\subset B_{n+1}$, and since $B_n \subset B_{n+1}$, there is an inner boundary component μ of B_{n+1} with K contained in the disk bounded by μ. Again, since $B_n \subset B_{n+1}$, $\mu \subset E$. By (4), A_n is contained in a disk in A_{n+1}; since $f(\mu) \subset f(E) \subset \operatorname{int}(M_{n+1})$, μ is of type (6). We have seen that no such curve of type (6) exists, and the lemma is proved.

If f: $\mathbf{R}^2 \to M$ is a proper map, then for any compact $C \subset M$ there is a complementary domain A of C and a disk $C \subset \mathbf{R}^2$ so that $f(\mathbf{R}^2 - D) \subset A$. If, for some choice of C, $[f|\partial D]$ is not trivial in $\pi_1(A)$, then f is an *essential* map. The following theorem will show, in particular, that a proper essential map may be replaced by a proper essential embedding.

THEOREM (2.2). *Let M be a 3-manifold and let f:$\mathbf{R}^2 \to M$ be a proper map. Suppose, for some compact $C \subset M$, that if $D \subset \mathbf{R}^2$ is the disk and A the complementary domain of C with $f(\mathbf{R}^2 - D) \subset A$, then $[f|\partial D] \notin G$ for G a normal subgroup of $\pi_1(A)$. Then for any neighborhood U of $f(\mathbf{R}^2)$ there is a proper embedding g: $\mathbf{R}^2 \to M$ and a disk $E \subset \mathbf{R}^2$ so that $g(\mathbf{R}^2) \subset U$, $g(\mathbf{R}^2 - E) \subset A$, and $[g|\partial E] \notin G$.*

PROOF. Choose a triangulation of M so that $U(f(\mathbf{R}^2)) \subset U$. We may replace M by $U(f(\mathbf{R}^2))$, C by $C \cap U(f(\mathbf{R}^2))$, A by the complementary domain A' of $C \cap U(f(\mathbf{R}^2))$ in $U(f(\mathbf{R}^2))$ which contains $f(\mathbf{R}^2 - D)$, and G by $i_*^{-1}(G)$, where i: $A' \to A$ is the inclusion map. Thus, it is sufficient to prove the theorem without the condition $g(\mathbf{R}^2) \subset U$.

Returning to the original notation we use Lemma (2.1) to find an exhausting sequence $\{M_i\}$ for M and a nested sequence of disks $\{D_i\}$ in $\mathbf{R}^2$ so that $C \subset M_1$, $D \subset D_1$, and D_i is a component of $f^{-1}(M_i)$. We assume that f is in general position with respect to ∂M_i for every i. Since $D \subset D_i$, it follows that $f(\partial D_i) \subset A$ for every i. As in [**2**] we may use Stallings' proof of the loop theorem [**5**] to choose a sequence of embedded disks $\{E_k\}$ in $M - \partial M$, and a sequence of simple loops $\{\lambda_j\}$ in A so that:

(1) $\bigcup_{k \geqslant i} \pi_0(\operatorname{Fr}(M_i) \cap E_k)$ is a finite set ($\pi_0(X)$ is the set of path components of X).

(2) $[\lambda_j] \notin G \subset \pi_1(A)$.

(3) If $k \geqslant j$ then λ_j is a component of $\operatorname{Fr}(M_j) \cap E_k$, and the disk in E_k bounded by λ_j contains λ_{j-1}.

For each $j < k$ we let $A(j, k)$ be the annulus in E_k bounded by λ_j and λ_{j+1}. Then for $i < j$ we set $J(i, j) = \{k |\ M_i \cap A(j, k) = \varnothing\}$.

LEMMA. *For each i there exist arbitrarily large integers m so that for all r, $\bigcap_{j=m}^{m+r} J(i, j) \neq \varnothing$. (Intuitively for fixed i annuli far enough out on E_k miss M_i.)*

PROOF. Let $m(i)$ be one greater than the number of elements in the finite set $\bigcup_{k \geqslant i} \pi_0(\mathrm{Fr}(M_i) \cap E_k)$. We first show that for fixed i any $m(i)$ of the sets $J(i, j)$ contain all but a finite number of the integers. Indeed, if $k \notin J(i, j)$, then $\mathrm{Fr}(M_i) \cap A(j, k) \neq \varnothing$. This cannot happen for $m(i)$ distinct values of j since $E_k \cap \mathrm{Fr}(M_i)$ has fewer than $m(i)$ components.

Suppose for some i the lemma is false. Then for all sufficiently large m, there exists r with $\bigcap_{j=m}^{m+r} J(i, j) = \varnothing$. Let $j(1) > i$ be sufficiently large and choose $r(1)$ so that $\bigcap_{j=j(1)}^{j(1)+r(1)} J(i, j) = \varnothing$. Inductively choose $j(s)$, $r(s)$, $s = 2, \ldots, m(i)$, so that $j(s) > j(s-1) + r(s-1)$, and $\bigcap_{j=j(s)}^{j(s)+r(s)} J(i, j) = \varnothing$. Let $k > j(m(i)) + r(m(i)) + 1$. Then for each $s = 1, 2, \ldots, m(i)$, there exists $j'(s)$, $j(s) < j'(s) \leqslant j(s) + r(s)$, so that $k \notin J(i, j'(s))$. Then $k \notin \bigcup_{s=1}^{m(i)} J(i, j'(s))$, which contradicts the assertion of the last paragraph. The lemma is proved.

Notice that if $k \in \bigcap_{j=m}^{m+r} J(i, j)$, then the annulus on E_k bounded by λ_m and λ_{m+r} does not meet M_i.

We will construct a singular plane in M whose only singularities are double curves. We realize this plane as the union of an embedded disk A_0 and embedded annuli A_i so that $A_i \cap A_j \neq \varnothing$ iff $|i - j| < 2$. Moreover, $A_i \cap \partial A_{i+1} = \partial A_i \cap A_{i+1}$ is a single component of the boundary of each.

Let A_0 be the disk bounded by λ_1 on E_2. Let $i(1)$ be an integer such that $A_0 \subset M_{i(1)}$. Let $m(1)$ be an integer such that, for all r, $\bigcap_{j=m(1)}^{m(1)+r} J(i(1), j)$ is not empty. The existence of $m(1)$ follows from our lemma.

Let A_1 be the annulus bounded by λ_1 and $\lambda_{m(1)}$ on $E_{m(1)+1}$. Let $i(2)$ be an integer such that $A_1 \subset M_{i(2)}$, and let $m(2) > m(1)$ be an integer such that for all r, $\bigcap_{j=m(2)}^{m(2)+r} J(i(2), j) \neq \varnothing$. Choose $k(2) \in \bigcap_{j=m(1)}^{m(2)} J(i(1), j)$, which is not empty by our choice of $m(1)$. Then let A_2 be the annulus on $E_{k(2)}$ bounded by $\lambda_{m(1)}$ and $\lambda_{m(2)}$. Observe that $A_2 \cap A_0 \subset A_2 \cap M_{i(1)} = \varnothing$.

Now we will inductively choose $i(s)$, $m(s)$, $k(s)$, and A_s. First we choose $i(s)$ so that $A_{s-1} \subset M_{i(s)}$. Then we choose $m(s) > m(s-1)$ so that, for all r, $\bigcap_{j=m(s)}^{m(s)+r} J(i(s), j) \neq \varnothing$, which is possible by our lemma. Next $k(s)$ is chosen in $\bigcap_{j=m(s-1)}^{m(s)} J(i(s), j)$, which is not empty by our choice of $m(s-1)$. Finally, A_s is the annulus on $E_{k(s)}$ bounded by $\lambda_{m(s-1)}$ and $\lambda_{m(s)}$. Notice that $A_{s-2} \cap A_s \subset M_{i(s-1)} \cap A_s = \varnothing$ by choice of $k(s)$.

We observe that a map $f'\colon \mathbf{R}^2 \to M$ which, in polar coordinates sends $\{(r, \theta) | n \leqslant r \leqslant n+1\}$ homeomorphically onto A_n, satisfies all the hypotheses on f in the statement of the theorem. Moreover, the singularities of f' are double curves in the interior of the annuli A_n. We show how to remove these singularities by cut and paste in order to obtain the proper embedding g.

For $n \geqslant 1$ let λ be a component of $\mathrm{int}(A_n) \cap \mathrm{int}(A_{n+1})$. Since a boundary component of A_n does not represent an element of G, it follows that λ is essential in A_n iff λ is essential in A_{n+1}. Each essential λ splits A_n into two annuli: an inner one which meets ∂A_{n-1}, and an outer one. For each $n \geqslant 1$ in turn, choose λ essential so that its inner annulus on A_n contains no further essential components of $\mathrm{int}(A_n) \cap \mathrm{int}(A_{n+1})$. Replace A_n by the inner annulus of λ on A_n, and A_{n+1} by the outer annulus of λ on A_{n+1}. For the case $n = 0$ we choose λ innermost on A_0, essential on

A_1. Replace A_1 by the outer annulus of λ on A_1, and A_0 by the disk bounded by λ on A_0. Now for every $n \geqslant 0$, $\operatorname{int}(A_n) \cap \operatorname{int}(A_{n+1})$ consists of simple closed curves, inessential on both.

For $n = 0, 1, 2, \ldots$ let $B_{2n} = A_{2n}$. Suppose λ is an innermost simple curve on B_{2n} in $A_{2n+1} \cap B_{2n}$. Then λ bounds a disk D on A_{2n+1}, and we may replace a slightly larger disk on A_{2n+1} in the usual way with a disk parallel to the disk bounded by λ on B_{2n}. After a finite number of such cuts we may assume that A_{2n+1} no longer meets B_{2n}. It is, however, possible that A_{2n+1} now meets A_{2n-1}, but only in a small neighborhood on B_{2n}. In this case we replace disks on A_{2n-1} by disks parallel to disks both on B_{2n} and A_{2n+1}. Thus A_{2n+1} misses both B_{2n} and A_{2n-1}. Finally, using cuts, A_{2n-1} can be assumed to miss B_{2n}. Now B_{2n-1} is the annulus resulting from A_{2n-1}, and $\bigcup_n B_n$ is a nonsingular plane. If $B_{2n+1} \cap M_i \neq \varnothing$, then $(B_{2n} \cup B_{2n+2}) \cap M_i \neq \varnothing$. Thus, $\bigcup_n B_n$ is properly embedded in M. There is a homeomorphism of $\mathbf{R}^2$ onto $\bigcup_n B_n$ which satisfies the conditions of the theorem.

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Department of Mathematics, Dartmouth College, Hanover, New Hampshire 03755

Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061