RING OF ENDOMORPHISMS OF A FINITE LENGTH MODULE

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Abstract. An example of a uniserial module $M_R$ of composition length 2, such that $S = \text{End}(M_R)$ acting on the left is not right artinian, is given. An elementary proof of a known result, that the ring of endomorphism of a finite length quasi-injective module $M_R$ acting on the left is left artinian, is also given.

Let $R$ be a ring with $1 \neq 0$ and $M$ an indecomposable unital right $R$-module of finite composition length. Let $S = \text{End}(M_R)$ be the ring of endomorphisms of $M$ acting on the left. A result due to Fitting (see Faith [1, Corollary 17.17']) gives that $S$ is a local ring with its Jacobson radical nilpotent. By taking $M = R_R$, for some local right artinian ring $R$, which is not left artinian, one immediately sees that $S$ need not be left artinian. No such simple example seems existing in literature, showing that $S$ need not be right artinian. In this note we give an example showing that $S$ need not be right artinian. It follows from [2, Proposition 1] that the ring of endomorphisms of a finite length quasi-injective module is left artinian; here we give an elementary proof of this result. For the terms and result used in this note we refer to Faith [1].

The following theorem provides the example.

Theorem 1. There exists a ring $R$ and a faithful right $R$-module $M$ such that

(i) $R$ is local, left artinian but not right artinian,
(ii) $M$ is a uniserial module having composition length 2,
(iii) $\text{End}(M_R) \approx R$.

Proof. Let $K$ be a field and $F = K(X_i)$ be the field of fractions over $F$, in an infinite set of indeterminates $\{X_i\}_{i \in \Lambda}$. Let $\sigma: F \to F$ be the $F$-endomorphism of the field $F$ such that $\sigma(X_i) = X_i^2$. Then

$$\sigma(F) = K(X_i^2) \neq F.$$  

A linear basis of $F$ over the subfield $\sigma(F)$ is $B = \{1\} \cup B_1$, where $B_1$ consists of all monomials of the form $X_{i_1}X_{i_2}\cdots X_{i_n}$ where $n \geq 1$ and $i_1, i_2, \ldots, i_n$ are finitely many distinct members of $\Lambda$. Consider the ring $R = F \times F$ in which addition is componentwise and the multiplication is by the rule

$$(\alpha, \beta)(\gamma, \delta) = (\alpha\gamma, \alpha\delta + \sigma(\gamma)\beta).$$

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$R$ is a local ring with maximal ideal $J = 0 \times F$. Further, $R$ has composition length 2, and $R$ is not right artinian. Let $G$ be the $\sigma(F)$-subspace of $F$ spanned by $B_1$. Then $A = 0 \times G$ is a right ideal of $R$. As $G$ is a maximal $\sigma(F)$-subspace of $F$, $M = R/A$ is a uniserial right $R$-module of composition length 2. Consider the idealizer of $A$,

$$I = I(A) = \{ r \in R : rA \subset A \}.$$

We have $S = \text{End}(M) = I/A$. It is easy to see that

$$I = \{(a, \beta) : aG \subset G \}.$$

We now show that $\sigma(F) = T$, where $T = \{ \alpha \in F : \alpha G \subset G \}$. Trivially $\sigma(F) \subset T$. Take any $\alpha \in F \setminus \sigma(F)$. Then $\alpha = \alpha_0 + \sum_{b \in B_1} \alpha_b b$ where $\alpha_0 \in \sigma(F)$, $\alpha_b \in \sigma(F)$, such that only finitely many of these coefficients are nonzero. As $\alpha \notin \sigma(F)$, for some $b \in B_1$, $\alpha_b \neq 0$. Fix a $b_0 \in B_1$ with $\alpha_{b_0} \neq 0$. Consider any $b \in B_1$ such that $b \neq b_0$. We can write

$$b_0 = cX_{j_1}X_{j_2} \cdots X_{j_r}, \quad b = cX_{k_1}X_{k_2} \cdots X_{k_s},$$

where $c$ is the largest degree monomial in $X_i$'s dividing $b_0$ and $b$ both. Then by definition of $B_1$, the indices $j_1, j_2, \ldots, j_r, k_1, k_2, \ldots, k_s$ are all distinct. Thus $c^2 \in \sigma(F)$ and

$$b_0b = c^2X_{j_1}X_{j_2} \cdots X_{j_r}X_{k_1}X_{k_2} \cdots X_{k_s} \in G.$$

Further as $0 \neq \alpha_{b_0}b_0^2 \in \sigma(F)$, we get

$$ab_0 = \alpha_{b_0}b_0^2 + \alpha_0b_0 + \sum_{b \neq b_0} \alpha_b (bb_0) \notin G.$$

Hence $\alpha \notin T$. This shows that $I = \sigma(F) \times F = \sigma(F) \times \sigma(F) + A$ and $I/A \simeq \sigma(F) \times \sigma(F) \simeq R$. Hence $S = \text{End}(M_R) \simeq R$. This proves the theorem.

We now give an elementary proof of the following result, which, otherwise, is a special case of [2, Proposition 1].

**Theorem 2.** Let $M_R$ be a quasi-injective module of finite composition length. Then $S = \text{End}(M_R)$ is left artinian.

**Proof.** We prove the result by induction on the composition length $d(M)$ of $M$. If $d(M) = 1$, $S$ is a division ring and the result holds. Let $d(M) > 1$ and the result hold for all quasi-injective modules of composition length less than $d(M)$. Let $J(M)$ be the Jacobson radical of $M$. Then $J(M)$ is quasi-injective and $d(J(M)) < d(M)$. So $T = \text{End}(J(M))$ is left artinian.

Define homomorphism $\sigma : S \to T$ such that $\sigma(f) = f|J(M)$ for every $f \in S$. As $M$ is quasi-injective $\sigma$ is onto. Thus $S/\text{Ker} \sigma \simeq T$. Now $M/J(M)$ is completely reducible. We write $\overline{M} = M/J(M) = K_1 \oplus K_2 \oplus \cdots \oplus K_n$. Let $K = \text{Socle}(M)$. Now $f \in \text{Ker} \sigma$ if and only if $f(J(M)) = 0$. As $M/J(M)$ is completely reducible, we get $f \in \text{Ker} \sigma$ if and only if $f(M) \subset K$. Thus

$$\text{Ker} \sigma = \text{Hom}(M, K) \approx \text{Hom}(\overline{M}, K) \approx \bigoplus_{i=1}^n \text{Hom}(K_i, K).$$
Consider any $K_i$. Consider any $f (\neq 0) \in \text{Hom}(K_i, K)$. Now $f$ is one-to-one and $K$ is completely reducible. Thus we can find $g: K \to K_i$ such that $gf = I_{K_i}$. So, for any $h \in \text{Hom}(K_i, K)$, $hgf = h$. Now $hg: K \to K$ can be extended to a member $\lambda$ of $S$. This all gives $\text{Hom}(K_i, K) = \text{Hom}(K_i, S K) = S[\text{Hom}(K_i, K)]$ is either simple or 0. Hence $S/\text{Ker } \sigma$ is a finite direct sum of simple modules. As $S/\text{Ker } \sigma$ is also left artinian, we get $S$ is left artinian.

**Remark** (added in proof). The module $M_R$ constructed in Theorem 1 is quasi-injective but not injective.

**References**