PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 94, Number 2, June 1985

RING OF ENDOMORPHISMS OF A FINITE LENGTH MODULE

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ABSTRACT. An example of a uniserial module M_R of composition length 2, such that $S = \mathrm{End}(M_R)$ acting on the left is not right artinian, is given. An elementary proof of a known result, that the ring of endomorphism of a finite length quasi-injective module M_R acting on the left is left artinian, is also given.

Let R be a ring with $1 \neq 0$ and M an indecomposable unital right R-module of finite composition length. Let $S = \mathrm{End}(M_R)$ be the ring of endomorphisms of M acting on the left. A result due to Fitting (see Faith [**1**, Corollary 17.17′]) gives that S is a local ring with its Jacobson radical nilpotent. By taking $M = R_R$, for some local right artinian ring R, which is not left artinian, one immediately sees that S need not be left artinian. No such simple example seems existing in literature, showing that S need not be right artinian. In this note we give an example showing that S need not be right artinian. It follows from [**2**, Proposition 1] that the ring of endomorphisms of a finite length quasi-injective module is left artinian; here we give an elementary proof of this result. For the terms and result used in this note we refer to Faith [**1**].

The following theorem provides the example.

THEOREM 1. *There exists a ring R and a faithful right R-module M such that*
(i) *R is local, left artinian but not right artinian*,
(ii) *M is a uniserial module having composition length* 2,
(iii) $\mathrm{End}(M_R) \approx R$.

PROOF. Let K be a field and $F = K(X_i)$ be the field of fractions over F, in an infinite set of indeterminates $\{X_i\}_{i\in\Lambda}$. Let $\sigma\colon F \to F$ be the K-endomorphism of the field F such that $\sigma(X_\alpha) = X_\alpha^2$. Then

$$\sigma(F) = K\left(X_i^2\right) \neq F.$$

A linear basis of F over the subfield $\sigma(F)$ is $B = \{1\} \cup B_1$, where B_1 consists of all monomials of the form $X_{i_1}X_{i_2}\cdots X_{i_n}$ where $n \geqslant 1$ and $i_1, i_2, \ldots, i_n$ are finitely many distinct members of Λ. Consider the ring $R = F \times F$ in which addition is componentwise and the multiplication is by the rule

$$(\alpha, \beta)(\gamma, \delta) = (\alpha\gamma, \alpha\delta + \sigma(\gamma)\beta).$$

Received by the editors May 22, 1984.
1980 *Mathematics Subject Classification*. Primary 16A65; Secondary 16A05.
Key words and phrases. Uniserial modules, ring of endomorphisms, quasi-injective, module, artinian ring, field of fractions.

R is a local ring with maximal ideal $J = 0 \times F$. Further ${}_R R$ has composition length 2, and R is not right artinian. Let G be the $\sigma(F)$-subspace of F spanned by B_1. Then $A = 0 \times G$ is a right ideal of R. As G is a maximal $\sigma(F)$-subspace of F, $M = R/A$ is a uniserial right R-module of composition length 2. Consider the idealizer of A,

$$I = I(A) = \{r \in R: rA \subset A\}.$$

We have $S = \operatorname{End}(M) \approx I/A$. It is easy to see that

$$I = \{(\alpha, \beta): \alpha G \subset G\}.$$

We now show that $\sigma(F) = T$, where $T = \{\alpha \in F: \alpha G \subset G\}$. Trivially $\sigma(F) \subset T$. Take any $\alpha \in F \setminus \sigma(F)$. Then $\alpha = \alpha_0 + \sum_{b \in B_1} \alpha_b b$ where $\alpha_0 \in \sigma(F)$, $\alpha_b \in \sigma(F)$, such that only finitely many of these coefficients are nonzero. As $\alpha \notin \sigma(F)$, for some $b \in B_1$, $\alpha_b \neq 0$. Fix a $b_0 \in B_1$ with $\alpha_{b_0} \neq 0$. Consider any $b \in B_1$ such that $b \neq b_0$. We can write

$$b_0 = cX_{j_1}X_{j_2} \cdots X_{j_r}, \qquad b = cX_{k_1}X_{k_2} \cdots X_{k_s},$$

where c is the largest degree monomial in X_i's dividing b_0 and b both. Then by definition of B_1, the indices $j_1, j_2, \dots, j_r, k_1, k_2, \dots, k_s$ are all distinct. Thus $c^2 \in \sigma(F)$ and

$$b_0 b = c^2 X_{j_1}X_{j_2} \cdots X_{j_r}X_{k_1}X_{k_2} \cdots X_{k_s} \in G.$$

Further as $0 \neq \alpha_{b_0} b_0^2 \in \sigma(F)$, we get

$$\alpha b_0 = \alpha_{b_0} b_0^2 + \alpha_0 b_0 + \sum_{b \neq b_0} \alpha_b (b b_0) \notin G.$$

Hence $\alpha \notin T$. This shows that $I = \sigma(F) \times F = \sigma(F) \times \sigma(F) + A$ and $I/A \approx \sigma(F) \times \sigma(F) \approx R$. Hence $S = \operatorname{End}(M_R) \approx R$. This proves the theorem.

We now give an elementary proof of the following result, which, otherwise, is a special case of [**2**, Proposition 1].

THEOREM 2. *Let M_R be a quasi-injective module of finite composition length. Then $S = \operatorname{End}(M_R)$ is left artinian.*

PROOF. We prove the result by induction on the composition length $d(M)$ of M. If $d(M) = 1$, S is a division ring and the result holds. Let $d(M) > 1$ and the result hold for all quasi-injective modules of composition length less than $d(M)$. Let $J(M)$ be the Jacobson radical of M. Then $J(M)$ is quasi-injective and $d(J(M)) < d(M)$. So $T = \operatorname{End}(J(M))$ is left artinian.

Define homomorphism σ: $S \to T$ such that $\sigma(f) = f|J(M)$ for every $f \in S$. As M is quasi-injective σ is onto. Thus $S/\operatorname{Ker}\sigma \approx T$. Now $M/J(M)$ is completely reducible. We write $\overline{M} = M/J(M) = K_1 \oplus K_2 \oplus \cdots \oplus K_n$. Let $K = \operatorname{Socle}(M)$. Now $f \in \operatorname{Ker}\sigma$ if and only if $f(J(M)) = 0$. As $M/J(M)$ is completely reducible, we get $f \in \operatorname{Ker}\sigma$ if and only if $f(M) \subset K$. Thus

$$\operatorname{Ker}\sigma = \operatorname{Hom}(M, K) \approx \operatorname{Hom}(\overline{M}, K) \approx \oplus \sum_{i=1}^{n} \operatorname{Hom}(K_i, K).$$

Consider any K_i. Consider any $f\,(\neq 0) \in \operatorname{Hom}(K_i, K)$. Now f is one-to-one and K is completely reducible. Thus we can find $g\colon K \to K_i$ such that $gf = I_{K_i}$. So, for any $h \in \operatorname{Hom}(K_i, K)$, $hgf = h$. Now $hg\colon K \to K$ can be extended to a member λ of S. This all gives $\operatorname{Hom}(K_i, K) = \operatorname{Hom}(K_i, {}_SK) = {}_S[\operatorname{Hom}(K_i, K)]$ is either simple or 0. Hence ${}_S\operatorname{Ker}\sigma$ is a finite direct sum of simple modules. As $S/\operatorname{Ker}\sigma$ is also left artinian, we get S is left artinian.

REMARK (added in proof). The module M_R constructed in Theorem 1 is quasi-injective but not injective.

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