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SET-THEORETIC COMPLETE INTERSECTIONS[1]

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ABSTRACT. In this article we establish that:

(1) Every monomial curve in $\mathbf{P}_k^n$ is a set-theoretic complete intersection, where k is a field of characteristic p (and thus generalize a result of R. Hartshorne [3]).

(2) Let k be an algebraically closed field of characteristic p and C a curve of $\mathbf{P}_k^n$. If there is a linear projection $\tau\colon \mathbf{P}_k^n \to \mathbf{P}_k^2$ with center of τ disjoint of C, $\tau(C)$ is birational to C and $\tau(C)$ has only cusps as singularities, then C is a set-theoretic complete intersection (and thus generalize a result of D. Ferrand [2]).

1. Let $(t^d, t^{a_1}u^{b_1},\dots,t^{a_{n-1}}u^{b_{n-1}}, u^d)$ with $n \geqslant 2$ be a parametrization of a curve C in $\mathbf{P}_k^n$, where $d = a_1 + b_1 = \cdots = a_{n-1} + b_{n-1}$ and k is a field of characteristic p. The curve C is said to be a *monomial curve*. Without losing generality we may assume that

$$d > a_1 > a_2 > \cdots > a_{n-1} > 0.$$

Let $\tau\colon k[x_1,\dots,x_n] \to k[t]$ be the mapping defined by

$$\tau(x_1) = t^d, \quad \tau(x_2) = t^{a_1},\dots,\tau(x_n) = t^{a_{n-1}}$$

and $P = \ker\tau$, $Q = P \cap k[x_1,\dots,x_{n-1}]$.

Given $f(x_1,\dots,x_n) \in k[x_1,\dots,x_n]$, let the highest homogeneous form of $f(x_1,\dots,x_n)$ be denoted by $\partial f(x_1,\dots,x_n)$. We shall establish

THEOREM 1. *There are binomials $f_1,\dots,f_{n-1} \in k[x_1,\dots,x_n]$ such that*

$$\sqrt{(f_1,\dots,f_{n-1})} = P, \tag{1}$$

$$\sqrt{(\partial f_1,\dots,\partial f_{n-1})} = (x_2,\dots,x_n). \tag{2}$$

PROOF. We shall use induction. The theorem is evident for $n = 2$. We may assume that there are binomials $f_1,\dots,f_{n-2} \in k[x_1,\dots,x_{n-1}]$ such that

$$\sqrt{(f_1,\dots,f_{n-2})} = Q, \tag{3}$$

$$\sqrt{(\partial f_1,\dots,\partial f_{n-2})} = (x_2,\dots,x_{n-1}). \tag{4}$$

We shall construct f_{n-1}. Let

$$g = \mathrm{GCD}(d, a_1,\dots,a_{n-2}), \quad e = \mathrm{GCD}(a_{n-1}, g), \quad a_{n-1} = a_{n-1}^* e, \quad g = g^* e.$$

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Then the positive semigroup generated by d, $a_1,\dots,a_{n-2}$ contains all large integers which are multiples of g. Hence there are integers m, $l_1,\dots,l_{n-1}$ such that

$$p^m g^* a_{n-1} = l_1 d + \sum_{i=2}^{n-1} l_i a_{i-1}.$$

In other words,

$$f_{n-1} = \left(x_n^{g^*}\right)^{p^m} - x_1^{l_1}\prod_{i=2}^{n-1} x_i^{l_i} \in P.$$

Clearly $p^m g^* > \sum_{i=1}^{n-1} l_i$. Hence we have

$$\partial f_{n-1} = \left(x_n^{g^*}\right)^{p^m} \quad \text{and} \quad \sqrt{(\partial f_1, \partial f_2,\dots,\partial f_{n-1})} = (x_2,\dots,x_n).$$

The conclusion (2) has been established. Let $h(x_1,\dots,x_n)$ be any weighted homogeneous polynomial in P with $x_1,\dots,x_n$ of weight $d,\dots,a_{n-1}$, respectively. Let

$$h(x_1,\dots,x_n) = \sum a_{i_1,\dots,i_n} x_1^{i_1}\cdots x_n^{i_n}.$$

Then $i_1 d + i_2 a_1 + \cdots + i_n a_{n-1} =$ const. It follows at once that

$$i_n a_{n-1} \equiv i_n' a_{n-1} \pmod g$$

for two indices i_n, i_n'. Moreover,

$$i_n \equiv i_n' \pmod{g^*}.$$

We conclude that

$$h(x_1,\dots,x_n) = x_n^c h^*\left(x_1,\dots,x_{n-1}, x_n^{g^*}\right)$$

and

$$h^*\left(x_1,\dots,x_{n-1}, x_n^{g^*}\right) \in P.$$

It is easy to see

$$\begin{aligned} h^*\left(x_1,\dots,x_{n-1}, x_n^{g^*}\right)^{p^m} &= \bar{h}\left(x_1,\dots,x_{n-1}, \left(x_n^{g^*}\right)^{p^m}\right) \\ &= h'(x_1,\dots,x_{n-1}) \pmod{f_{n-1}} \end{aligned}$$

and

$$h'(x_1,\dots,x_{n-1}) \in P \cap k[x_1,\dots,x_{n-1}] = Q.$$

Putting the above arguments together we get $h \in \sqrt{(f_1,\dots,f_{n-1})}$. Q.E.D.

COROLLARY. *Let $\mathbf{A}_k^n \subset \mathbf{P}_k^n$ by dehomogenizing $x_{n+1} = 1$. For the monomial curve C this means setting $u = 1$. The binomials $(f_1,\dots,f_{n-1})$ define the affine piece of C and only pass through the point $(1,0,\dots,0)$ at ∞. Hence the homogenized polynomials $f_1^{\#},\dots,f_{n-1}^{\#}$ define the projective curve C.*

2. Let C be a curve in $\mathbf{P}_k^n$, where k is an algebraically closed field of characteristic p. Let τ be a linear projection $\mathbf{P}_k^n \to \mathbf{P}_k^2$ with center D. Suppose $D \cap C = \varnothing$, C is birational to $\tau(C)$, and $\tau(C)$ has only cusps as singularities. Then we have

THEOREM 2. *Under the above assumption, C is a set-theoretic complete intersection.*

PROOF. Choose $L_\infty \subset \mathbf{P}_k^2$ so that $L_\infty \cap C'$ consists of d distinct points where $C' = \tau(C)$ and d = degree of C' = order of C. Take L_∞ as the line at ∞, the complement of $L_\infty = \mathbf{A}_k^2$. Let $f(x_1, x_2)$ = the defining equation of $C' \cap \mathbf{A}_k^2$. Then $d = \deg f(x_1, x_2)$ and $\partial f(x_1, x_2)$ = highest homogeneous form has d distinct roots.

Let H be the hyperplane spanned by L_∞ and D. The hyperplane $H = \tau^{-1}(L_\infty)$. Consider H as the hyperplane at ∞ and $\mathbf{P}_k^n = \mathbf{A}_k^n \cup H$. Let the coordinate ring $\mathbf{A}_k^n = k[x_1, x_2, \ldots, x_n]$.

The condition that C' has only cusps as singularities implies that the preimage $\tau^{-1}(P)$ consists of a single point $\{Q\}$ for every $P \in C'$.

We shall discuss the corresponding problem for the affine pieces $C \cap \mathbf{A}_k^n \xrightarrow{\tau} C' \cap \mathbf{A}_k^2$ first. Let I = the ideal of $C \cap \mathbf{A}_k^n$, $J = (f(x_1, x_2))$. Let E be the conductor of $k[x_1, x_2]/J \hookrightarrow k[x_1, x_2, \ldots, x_n]/I$. Let $Q_1, \ldots, Q_m$ be the points belonging to E on $C \cap \mathbf{A}_k^n$ and $P_1, \ldots, P_m$ be the corresponding points on $C' \cap \mathbf{A}_k^2$. Then at $Q_1, \ldots, Q_n$ let $\bar{x}_i$ be the canonical image of x_i in $k[x_1, \ldots, x_n]/I$. We have

$$\bar{x}_i^{p^{m_i}} \in (k[x_1, x_2]/J)_{P_j} \quad \forall i, j$$

for m_i large enough. Hence we get

$$\bar{x}_i^{p^{m_i}} \in k[x_1, x_2]/J \quad \forall i.$$

In other words, there are m_i and $g_i(x_1, x_2)$ such that

$$f_i = x_i^{p^{m_i}} - g_i(x_1, x_2) \in I \quad \forall i \geqslant 3.$$

We claim that $(f, f_3, \ldots, f_n)$ defines $C \cap \mathbf{A}_k^n$. We follow Cowsik-Nori. Let $g \in I$ and $N \geqslant m_i$ $\forall i$. Then clearly

$$\begin{aligned} g(x_1, x_2, \ldots, x_n)^{p^N} &= \bar{g}\left(x_1, x_2, x_3^{p^N}, \ldots, x_n^{p^N}\right) \\ &= g'(x_1, x_2) \pmod{f_3, \ldots, f_n} \end{aligned}$$

and

$$g'(x_1, x_2) \in J = (f)$$

so that we have $g \in \sqrt{(f, f_3, \ldots, f_n)}$.

To finish the proof of the projective case we homogenize $f_1, f_3, \ldots, f_n$ to $f^\#$, $f_3^\#, \ldots, f_n^\#$. Let the d points at ∞ be $\bar{Q}_1, \ldots, \bar{Q}_d$ and $\bar{P}_1, \ldots, \bar{P}_d$ for $\mathbf{P}_k^n$ and $\mathbf{P}_k^2$, respectively. We claim that $f, f_3, \ldots, f_n$ can be modified so that $(\partial f, \partial f_3, \ldots, \partial f_n)$ only pass through $\bar{Q}_1, \ldots, \bar{Q}_d$. There are two cases to be considered: *Case* I. $\deg f_i > p^{m_i}$. *Case* II. $\deg f_i = p^{m_i}$. In Case I, since f_i indeed passes through $\bar{Q}_1, \ldots, \bar{Q}_d$, we conclude at once that $g_i(x_1, x_2)$ passes through $\bar{P}_1, \ldots, \bar{P}_d$. In other words

$$\partial f(x_1, x_2) | \partial g_i(x_1, x_2).$$

So we may modify $g_i(x_1, x_2)$ by $f(x_1, x_2)$ to reduce the degree of f_i. Finally we may assume we are in Case II. Now the polynomial ∂f_i is an inseparable equation in x_i for $i \geqslant 3$. So there is only one possible solution for x_i if the values of x_1, x_2 are fixed. Thus there is only one point lying over $\bar{P}_1, \ldots, \bar{P}_d$, namely, $\bar{Q}_1, \ldots, \bar{Q}_d$. Q.E.D.

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