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TRANSLATION-INVARIANT LINEAR FORMS ON $L_p(G)$

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ABSTRACT. Let G be a compact group such that the identity representation of G is not contained in the regular representation on $L_2^0(G, \lambda_G)$ of G with the discrete topology. Then any left translation invariant linear form on $L_p(G)$, $1 < p < \infty$, is continuous and must be a constant times the Haar integral. This shows that many classical matrix groups G admit only continuous left translation invariant linear forms on $L_p(G)$, $1 < p < \infty$.

Let G be a compact Hausdorff group and let λ_G be the normalized Haar measure on G. A *linear form* on $L_p(G)$ is a functional φ on $L_p(G)$ which is linear. Given $g \in G$ and $f \in L_p(G)$, define ${}_gf(x) = f(g^{-1}x)$ for a.e. $x \in G$. We say that the linear form φ is *invariant* if $\varphi({}_gf) = \varphi(f)$ for all $g \in G$. Under what conditions on G is a linear form on $L_p(G)$ automatically continuous? This problem has received considerable attention for a variety of groups and function spaces. If G is a connected compact abelian group, Meisters and Schmidt **[6]** showed that any invariant linear form on $L_2(G)$ is continuous. This was recently extended by Bourgain **[1]** to $L_p(T)$, $1 < p < \infty$. On the other hand, Meisters **[5]** has shown that some totally-disconnected compact groups have discontinuous invariant linear forms on $L_p(G)$. The examples given here are for quite different groups than have been studied previously in this context.

We say G has the *mean-zero weak containment property* if for all $g_1, \dots, g_n \in G$, and $\varepsilon > 0$, there exists $f \in L_2^0(G) = \{f \in L_2(G): \int f\, d\lambda_G = 0\}$ such that $\|f\|_2 = 1$ and $\|{}_{g_i}f - f\|_2 < \varepsilon$ for $i = 1, \dots, n$. That is, the identity representation of G is contained in the regular representation on $L_2^0(G, \lambda_G)$ of G with the discrete topology. If G is amenable as a discrete group, then G has this property. On the other hand, if G contains a dense discrete subgroup with Kahzdan's property T, then G does not have the mean-zero weak containment property. Recently, in solving the Banach-Ruziewicz problem for S^2 and S^3, V. G. Drinfeld **[3]** has shown that SO(3) and SO(4) do not have the mean-zero weak containment property. This, together with Margulis **[4]**, shows that SO(n), $n \geqslant 3$, does not have the mean-zero weak containment property. Moreover, it follows from **[3, 4]** that any compact simple Lie group does not have the mean-zero weak containment property. See **[2, 7]** for a

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discussion of this property and its relationship to the uniqueness of invariant means on $L_\infty(G)$.

Lemma. *Suppose G does not have the mean-zero weak containment property. Then there exists $g_1,\dots,g_n \in G$ such that for some $\delta_p < 1$ and any $f \in L_p^0(G)$, $1 < p < \infty$, we have*

$$\left\| \frac{1}{n+1}\left(f + \sum_{i=1}^{n} \delta_{g_i} f \right) \right\|_p \leqslant \delta_p \|f\|_p.$$

Proof. Let $\mu = (1/(n+1))(\delta_e + \sum_{i=1}^n \delta_{g_i})$ where e is the identity in G and δ_g, $g \in G$, denotes the Dirac mass measure at g. Then μ acts by convolution on $L_p^0(G)$ with $\|\mu\|_{L_p^0} \leqslant 1$. Suppose $\|\mu\|_{L_2^0} = 1$. Then there exists a sequence $(f_m) \subset L_2^0(G)$ such that $\|f_m\|_2 = 1$ for all $m \geqslant 1$, and $\lim_{m\to\infty}\|\mu * f_m\|_2 = 1$. It is easy to show, as in [**2**, Theorem 1.1], that this forces

$$\lim_{m\to\infty} \left\| {}_{g_i}f_m - f_m \right\|_2 = 0 \quad \text{for each } i = 1,\dots,n.$$

So if G does not have the mean-zero weak containment property, there exists μ as above with $\|\mu\|_{L_2^0} < 1$. An interpolation argument as in [**8**] shows $\|\mu\|_{L_p^0} < 1$ for all p, $1 < p < \infty$. □

Proposition. *Suppose G does not have the mean-zero weak containment property. Then there exists $g_1,\dots,g_n \in G$ such that for every $f \in L_p(G)$, $1 < p < \infty$, there exists $h \in L_p^0(G)$ such that*

$$f = \left(\int f \, d\lambda_G \right) 1_G + \sum_{i=1}^{n} \left(h - {}_{g_i}h \right).$$

Proof. Let μ be as in the proof of the lemma. Denote by μ^n, $n \geqslant 1$, the nth-convolution power of μ, and let $\mu^0 = \delta_e$. Let $f_0 \in L_p^0(G)$. Since $\|\mu\|_{L_p^0} < 1$, the series $k = \sum_{n=1}^\infty \mu^n * f_0$ converges in $L_p^0(G)$. Also, $k - \mu * k = \mu^0 * f_0 = f_0$. If $h = k/(n+1)$, then

$$f_0 = \frac{nk}{n+1} - \frac{1}{n+1}\sum_{i=1}^{n} {}_{g_i}k = nh - \sum_{i=1}^{n} {}_{g_i}h = \sum_{i=1}^{n} \left(h - {}_{g_i}h \right).$$

If $f \in L_p(G)$, let $f_0 = f - (\int f \, d\lambda_G)1_G$ to get the representation of the proposition. □

Theorem. *Suppose G does not have the mean-zero weak containment property. Then there exists $g_1,\dots,g_n \in G$ such that any linear form on $L_p(G)$, $1 < p < \infty$, invariant under $g_1,\dots,g_n$ must be continuous and therefore a scalar times the Haar integral.*

Proof. Let $g_1,\dots,g_n \in G$ as in the proposition. Let φ be a linear form invariant under $g_1,\dots,g_n$. Then, using the representation of the proposition, $\varphi(f) = \varphi(1_G)\int f \, d\lambda_G$. □

Remark 1. Suppose G is abelian and H is a countable subgroup of G. Then [**8**, Theorem 14] shows that

$$S = \operatorname{span}\left\{ {}_g f - f \colon g \in H, f \in L_p(G) \right\}, \qquad 1 < p < \infty,$$

is not closed. Hence, there exists discontinuous H invariant linear forms on $L_p(G)$, $1 < p < \infty$. This shows that the result of Meisters and Schmidt [**6**] requires the full invariance under G, and contrasts the result above with previous results in that the invariance hypothesis is weakened.

REMARK 2. A theorem similar to the above is true for an ergodic group action on a probability space where the action does not have the mean-zero weak containment property. See Proposition 11 and the remarks after Proposition 13 in [**8**]. For example, let φ be a linear form on $L_p(T^2)$, $1 < p < \infty$, where T denotes the circle group. Let τ_1, τ_2: $T^2 \to T^2$ be defined by $\tau_1(z_1, z_2) = (z_2, z_1)$, $\tau_2(z_1, z_2) = (z_1 z_2, z_2)$ for all $z_1, z_2 \in T$. If φ is a linear form on $L_p(T^2)$, $1 < p < \infty$, such that $\varphi(f \circ \tau_i) = \varphi(f)$ for all $f \in L_p(T^2)$ and $i = 1, 2$, then φ is continuous and a scalar times the Haar integral.

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