

STRONG LIMIT THEOREMS FOR ORTHOGONAL SEQUENCES IN VON NEUMANN ALGEBRAS

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ABSTRACT. Let A be a von Neumann algebra with a faithful normal state ϕ . It is shown that if a sequence (x_n) in A is orthogonal relative to ϕ and satisfies the condition

$$\sum_k \phi(|x_k|^2) \left(\frac{\log k}{k} \right)^2 < \infty,$$

then $\frac{1}{n} \sum_{k=1}^n x_k \rightarrow 0$ almost uniformly in A . Some other results related to this theorem are also discussed.

1. Preliminaries. Strong limit theorems in a von Neumann algebra with a faithful normal state, close to the classical ones, have been obtained for the ergodic averages and martingales (see, for example, [4–7]). To the best of our knowledge there are only very few results concerning sequences whose elements are orthogonal (or uncorrelated or independent) relative to a state: for uniformly bounded sequences [3, 8] and for sequences satisfying some rather special conditions of Rosenblatt's type [5]. Let us notice here that many results similar to the classical ones are known for traces, i.e. for finite von Neumann algebras [3, 8]. However, for states the proofs are more complicated and, as a rule, need some new approach because the states, in general, are not subadditive on the lattice of projections in a von Neumann algebra. In case of a noncommutative von Neumann algebra, the notion of independence of its subalgebras with respect to a state [3] seems to be very restrictive, especially when the state is not tracial. On the other hand, some ideas of the correlation theory of stochastic processes can be developed in a natural way on the ground of quantum mechanics [1]. In particular, it seems to be important to have some information about the asymptotic behaviour of sequences of uncorrelated observables. This is the motivation of our work. The main goal of this paper is to prove a strong law of large numbers for sequences orthogonal relative to a state. Let us begin with some notation and definitions. Throughout, A will denote a von Neumann algebra with a faithful normal state Φ . For a projection p , always $p^\perp = 1 - p$, and $|x|^2 = x^*x$ for

Received by the editors June 8, 1984 and, in revised form, August 16, 1984.

1980 *Mathematics Subject Classification*. Primary 46L10, 81C20, 60F15.

Key words and phrases. von Neumann algebras, faithful normal state, orthogonal sequences, independent sequences, almost uniform convergence, almost complete convergence, strong law of large numbers.

¹On leave from Lodz University, Poland. This paper was written while the author held a visiting position at the University of Tennessee, whose hospitality he gratefully acknowledges.

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every $x \in A$. We say that a sequence (x_n) in A converges almost uniformly to x if, for each $\epsilon > 0$, there is a projection $p \in A$ with $\phi(p^\perp) < \epsilon$ and such that

$$\|(x_n - x)p\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

A sequence (x_n) in A is said to be convergent almost completely to x if, for every $\epsilon > 0$, there exists a sequence (q_n) of projections in A such that $\sum_n \phi(1 - q_n) < \infty$ and $\|(x_n - x)q_n\| < \epsilon$ for $n = 1, 2, \dots$. Relations between the almost complete and almost uniform convergence will be discussed in the Appendix.

Two elements x and y in A are said to be orthogonal relative to ϕ if $\phi(x^*y) = 0$. Two elements x and y are said to be uncorrelated (relative to ϕ) if $\phi(xy) = \phi(x)\phi(y)$. Of course, both relations are symmetric (it is enough to take the adjoints).

Obviously, if x and y are uncorrelated, then the “centered” elements $x - \phi(x)$ and $y - \phi(y)$ are orthogonal. For a selfadjoint operator x in A and a Borel set Z on the real line, the symbol $e_Z(x)$ will denote the spectral projection of x corresponding to the set Z ; in particular,

$$x = \int_{-\infty}^{\infty} \lambda e_{d\lambda}(x) \quad (\text{spectral representation}).$$

In the sequel we shall need the following theorem which is a special case of the noncommutative maximal ergodic theorem of Goldstein [5].

THEOREM 1. *Let (a_n) be a sequence of positive elements in A and let (ϵ_k) be a sequence of positive numbers such that*

$$\sum_{k=1}^{\infty} \epsilon_k^{-1} \phi(a_k) < \frac{1}{2}.$$

Then there exists a projection $p \in A$ with $\phi(p) \geq 1 - \sum_{k=1}^{\infty} \epsilon_k^{-1} \phi(a_k)$ and such that $\|pa_n p\| < 2\epsilon_n$ for $n = 1, 2, \dots$

2. Strong law of large numbers. In this section we prove the following strong limit theorem on sequences orthogonal relative to a state.

THEOREM 2. *Let A be a von Neumann algebra with a faithful normal state ϕ and let (x_n) be a sequence of pairwise orthogonal elements of A (i.e. $\phi(x_n^*x_m) = 0$ for $n \neq m$). If*

$$(1) \quad \sum_{n=1}^{\infty} \left(\frac{\log n}{n} \right)^2 \phi(|x_n|^2) < \infty,$$

then the averages

$$(2) \quad S_n = \frac{1}{n} \sum_{k=1}^n x_k$$

converge to zero almost uniformly.

In order to prove this theorem we start with the following

PROPOSITION. *Let (y_n) be a sequence of pairwise orthogonal elements of A . Put*

$$(3) \quad t_n = \sum_{k=1}^n y_k.$$

Then there exists in A a sequence (B_m) of positive operators such that

$$(4) \quad |t_n|^2 \leq (m + 1)B_m \quad \text{for } 1 \leq n \leq 2^m,$$

and

$$(5) \quad \phi(B_m) \leq (m + 1) \sum_{k=1}^{2^m} \phi(|y_k|^2).$$

PROOF. The proof is based on the idea which goes back to Plancherel [10] and is well known in the theory of orthogonal series [2]. Namely, we start from the dyadic representation of the index n . Divide the interval $I = (0, 2^m]$ into intervals $(0, 2^{m-1}]$ and $(2^{m-1}, 2^m]$, each of these intervals into halves, and so on; we obtain in this way a sequence of partitions of I . The elements of the first partition are of length 2^{m-1} , the elements of the r th partition are of length 2^{m-r} . For a positive integer $n \leq 2^m$, we take its dyadic representation. Then the interval $(0, n]$ can be written as the sum of at most m disjoint intervals $I_j^{(n)}$, each of which belongs to a different partition, that is,

$$(6) \quad (0, n] = \bigcup_{j=0}^m I_j^{(n)},$$

where $I_j^{(n)}$ is empty or of length $|I_j^{(n)}| = 2^j$ ($j = 1, 2, \dots, m$). We can write

$$(7) \quad t_n = \sum_{j=0}^m \sum_{k \in I_j^{(n)}} y_k$$

(of course, we put $\sum_{k \in I_j^{(n)}} y_k = 0$ in the case $I_j^{(n)}$ is empty). Let us remark now that for any sequence z_1, z_2, \dots, z_n of elements of A we have

$$(8) \quad \left| \sum_{k=1}^n z_k \right|^2 \leq n \sum_{k=1}^n |z_k|^2.$$

This easily follows by induction from the inequality $x*y + y*x \leq x*x + y*y$. Put

$$(9) \quad B_m = \sum_I \left| \sum_{k \in I} y_k \right|^2,$$

where I runs over all intervals which appear as the elements of the partitions of $(0, 2^m]$. Then we have

$$(10) \quad |t_n|^2 \leq (m + 1) \sum_{j=0}^m \left| \sum_{k \in I_j^{(n)}} y_k \right|^2 \leq (m + 1)B_m.$$

Moreover, B_m does not depend on $n \in (0, 2^m]$ and (5) holds, which completes the proof of the proposition.

PROOF OF THEOREM 2. Put $S_N = (1/N)\sum_{k=1}^N x_k$. Let $2^k < N \leq 2^{k+1}$. Then

$$(11) \quad |S_N - S_{2^k}|^2 = \left| \left(\frac{1}{N} - \frac{1}{2^k} \right) \sum_{s=1}^{2^k} x_s + \frac{1}{N} \sum_{s=2^k+1}^N x_s \right|^2 \\ \leq 2 \left[\left(\frac{1}{N} - \frac{1}{2^k} \right)^2 \left| \sum_{s=1}^{2^k} x_s \right|^2 + \frac{1}{N^2} \left| \sum_{s=2^k+1}^N x_s \right|^2 \right].$$

Applying the Proposition, we have that

$$|S_n - S_{2^k}|^2 \leq 2^{1-2k} \left[\left| \sum_{s=1}^{2^k} x_s \right|^2 + (k + 2) B_k \right],$$

where B_k is a positive operator, independent on $N \in (2^k, 2^{k+1}]$ and such that

$$(12) \quad \phi(B_k) \leq (k + 2) \sum_{s=2^k+1}^{2^{k+1}} \phi(|x_s|^2);$$

thus, for $2^k < N \leq 2^{k+1}$, we have

$$(13) \quad |S_N - S_{2^k}|^2 \leq D_k,$$

where $D_k \in A_+$, and

$$(14) \quad \phi(D_k) \leq 2^{1-2k} \left[\sum_{s=1}^{2^k} \phi(|x_s|^2) + (k + 2) \sum_{s=2^k+1}^{2^{k+1}} \phi(|x_s|^2) \right].$$

By the assumptions of the theorem, we have

$$(15) \quad \begin{aligned} \sum_{k=1}^{\infty} \phi(D_k) &\leq \sum_{k=1}^{\infty} 2^{1-2k} \frac{2^{2k}}{k^2} \sum_{s=1}^{2^k} \phi(|x_s|^2) \left(\frac{\log s}{s} \right)^2 \\ &\quad + \sum_{k=1}^{\infty} 2^{1-2k} \frac{2^{2k}}{k^2} (k + 2)^2 \sum_{s=2^k+1}^{2^{k+1}} \phi(|x_s|^2) \left(\frac{\log s}{s} \right)^2 \\ &\leq \sum_{s=1}^{\infty} \phi(|x_s|^2) \left(\frac{\log s}{s} \right)^2 \left(\sum_{k=1}^{\infty} \frac{2}{k^2} + \text{const.} \right) \end{aligned}$$

Moreover,

$$(16) \quad \begin{aligned} \sum_{k=1}^{\infty} \phi(|S_{2^k}|^2) &= \sum_{k=1}^{\infty} \frac{1}{2^{2k}} \sum_{s=1}^{2^k} \phi(|x_s|^2) \\ &\leq \sum_{k=1}^{\infty} 2^{-2k} \frac{2^{2k}}{k^2} \sum_{s=1}^{\infty} \phi(|x_s|^2) \left(\frac{\log s}{s} \right)^2 < \infty. \end{aligned}$$

Let $\varepsilon > 0$ be given. By (15) and (16) one can find a sequence (ε_k) of positive numbers such that $\varepsilon_k \rightarrow 0$ and

$$(17) \quad \sum_{k=1}^{\infty} \varepsilon_k^{-1} \phi(|S_{2^k}|^2 + D_k) < \frac{\varepsilon}{2}.$$

By Theorem 1, there is a projection $p \in A$ with $\phi(p) \geq 1 - \varepsilon$ and such that

$$(18) \quad \|p|S_{2^k}|^2 p\| < 2\varepsilon_k \quad \text{and} \quad \|pD_k p\| < 2\varepsilon_k.$$

Thus, for $2^k < N \leq 2^{k+1}$, we have the estimation

$$\begin{aligned} \|S_N p\|^2 &= \|(S_N - S_{2^k}) p + S_{2^k} p\|^2 \\ &\leq 2 \left[\|(S_N - S_{2^k}) p\|^2 + \|S_{2^k} p\|^2 \right] \\ &= 2 \left[\|p|S_N - S_{2^k}|^2 p\| + \|p|S_{2^k}|^2 p\| \right] \\ &\leq 2 \left[\|pD_k p\| + \|p|S_{2^k}|^2 p\| \right] < 8\varepsilon_k \rightarrow 0 \quad \text{as } N \rightarrow \infty. \end{aligned}$$

This means that $S_N = (1/N)\sum_{s=1}^N x_s$ converges to zero almost uniformly, which ends the proof.

3. Comments. Let us compare Theorem 2 with the classical results. The Rademacher-Menchoff theorem on the almost sure convergence of orthogonal series [2] gives, via Kronecker's lemma, the following strong law of large numbers.

If, for a sequence (X_n) of uncorrelated random variables,

$$\sum_n \left(\frac{\log n}{n}\right)^2 \text{var}(X_n) < \infty,$$

then $(1/n)\sum_{k=1}^n (X_k - EX_k) \rightarrow 0$ with probability one.

Of course, Theorem 2 can (and should) be treated as the noncommutative extension of the strong law of large numbers just formulated.

Under some conditions stronger than in Theorem 2 one can obtain better convergence of the averages. Namely, it is easy to prove the following two theorems.

THEOREM 3. *Let (x_n) be a sequence in A orthogonal relative to a state ϕ . If*

$$\sum_{s=1}^{\infty} a_s \phi(|x_s|^2) < \infty \quad \text{when } 0 < a_s \downarrow 0,$$

and

$$\sum_{s=1}^{\infty} \frac{1}{s^2 a_s} < \infty,$$

then $(1/n)\sum_{s=1}^n x_s \rightarrow 0$ almost completely.

PROOF. Put $s_N = N^{-1}\sum_{s=1}^N x_s$. Then

$$\phi(|S_N|^2) = \frac{1}{N^2} \sum_{s=1}^N \phi(|x_s|^2) \leq \frac{1}{N^2 a_N} \sum_{s=1}^N a_s \phi(|x_s|^2),$$

thus $\sum \phi(|S_N|^2) < \infty$. For $\epsilon > 0$, let us put $q_N = e_{[0, \epsilon^2]}(|S_N|^2)$. Then $\|S_N q_N\| < \epsilon$ for $N = 1, 2, \dots$. Moreover,

$$\sum_{N=1}^{\infty} \phi(q_N^\perp) \leq \epsilon^{-2} \sum_{k=1}^{\infty} \phi(|S_k|^2) < \infty,$$

which ends the proof.

The next theorem is a stronger version of Batty's result [3, Theorem 4.1].

THEOREM 4. *Let (X_n) be a uniformly bounded weakly independent (in the sense of [3]) sequence in A , with $\phi(X_k) = 0$. Then $(1/n)\sum_{s=1}^n X_s \rightarrow 0$ almost completely.*

PROOF. Let $\epsilon > 0$. Put $S_N = (1/N)\sum_{s=1}^N X_s$. It is easy to show that $\phi(|S_N|^4) \leq N^{-4}(3N^2 - N)$, hence $\sum_N \phi(|S_N|^4) < \infty$. Put $q_N = e_{[0, \epsilon^4]}(|S_N|^4)$. It follows that $\sum_N \phi(q_N^\perp) < \infty$ and $\|S_N q_N\|^4 \leq \| |S_N|^4 q_N \| < \epsilon^4$ for $N = 1, 2, \dots$, which ends the proof.

4. Appendix. In this section we discuss the relationship between the almost uniform and almost complete convergence. Let us remark first that if ϕ is a trace, then the almost complete convergence implies the almost uniform convergence.

Indeed, let $X_n \rightarrow 0$ almost completely. Then there is a sequence of projections q_n such that $\|X_n q_n\| < \varepsilon$ for $n = 1, 2, \dots$, and $\sum_n \phi(q_n^\perp) < \infty$. Putting $p_n = \bigwedge_{s=n}^\infty q_s$, we have $\phi(1 - p_n) \leq \sum_{s=n}^\infty \phi(1 - q_s) \rightarrow 0$. The last condition implies the almost uniform convergence [9, Theorem 3.1].

When ϕ is a state, we have the following

THEOREM 5. *Let A be a von Neumann algebra with a faithful normal state ϕ . Let (X_n) be a bounded sequence in A . If $X_n \rightarrow X$ almost completely, then $X_n \rightarrow X$ almost uniformly.*

PROOF. We can assume that $\|X_n\| \leq 1$ and $x = 0$. Let $\varepsilon > 0$. We find a sequence (q_n) of projections in A such that $\sum_n \phi(q_n^\perp) < \infty$ and $\|X_n q_n\| < \varepsilon$ for $n = 1, 2, \dots$. Let us fix a sequence (ε_n) of positive numbers such that $\varepsilon_n \rightarrow 0$ and $\sum_{n=1}^\infty \phi(1 - q_n) \varepsilon_n^{-1} < \varepsilon/2$. By Theorem 1 there is a projection $p \in A$ with $\phi(p^\perp) < \varepsilon$ and such that $\|p q_n^\perp p\| = \|q_n^\perp p\|^2 < 2\varepsilon_n$ for $n = 1, 2, \dots$. Then we have

$$\begin{aligned} \|X_n p\| &\leq \|X_n q_n p\| + \|X_n q_n^\perp p\| \leq \|X_n q_n\| + \|q_n^\perp p\| \\ &\leq \varepsilon + (2\varepsilon_n)^{1/2} < 2\varepsilon \quad \text{for } n > M_0(\varepsilon). \end{aligned}$$

Thus the following condition is satisfied:

- (*) For each $\varepsilon > 0$ there is a projection p with $\phi(p) \geq 1 - \varepsilon$ and such that $\|X_n p\| < \varepsilon$ for $n > n_0(\varepsilon)$.

We shall show that the condition (*) implies the almost uniform convergence of x_n to zero. We first prove that (*) implies:

- (**) For each $\varepsilon \geq 0$ and for each projection $q \in A$ there is a projection $r \in A$ such that $r \leq q$, $\phi(q - r) < \varepsilon$ and $\|X_n r\| < \varepsilon$ for n large enough.

Indeed, let $0 < \varepsilon_n \rightarrow 0$. By (*) we find in A a sequence of projections r_n with $\phi(r_n) > 1 - \varepsilon_n$, and a sequence of positive integers $m(n)$ such that $\|X_m r_n\| < \varepsilon_n$ for $m > m(n)$. Let a projection $q \in A$ be given. Then $\phi(q r_n^\perp q) \rightarrow 0$ and we can fix n_0 such that $\varepsilon_{n_0} < \varepsilon$ and $\phi(q r_{n_0}^\perp q) < \varepsilon^4$. Let $r = q e_{[0, \varepsilon^2]}(q r_{n_0}^\perp q)$. Then we have $r \leq q$, $\phi(q - r) < \varepsilon^2$ and $\|r_{n_0}^\perp r\| < \varepsilon$. Moreover, $\|X_m r_{n_0}\| < \varepsilon_{n_0}$ for $m > m(n_0)$. Thus $\|X_m r\| \leq \|X_m r_{n_0}^\perp r\| + \|X_m r_{n_0}\| < 2\varepsilon$ for $m > m(n_0)$, which ends the proof of the implication (*) \rightarrow (**).

Let $\varepsilon > 0$ be given. By (**) we can find in A a sequence of projections (p_n) such that $1 = p_0 \geq p_1 \geq p_2 \geq \dots$, $\phi(p_n - p_{n+1}) < 2^{-n}\varepsilon$ and $\|X_m p_n\| < 2^{-n}\varepsilon$ for $m > m(n_0)$. Put $p = \bigwedge_k p_k$. Then $\phi(p^\perp) = \sum_n \phi(p_n - p_{n+1}) < \varepsilon$. Moreover, $\|X_m p\| \leq \|X_m p_{n_0}\| < 2^{-n_0}\varepsilon$ for $m > m(n_0)$. That means that $X_m \rightarrow 0$ almost uniformly. The proof is completed.

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