REMARKS ON A PAPER BY A. AZIZ

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Abstract. The paper consists of two parts. In the first part a short proof of the main theorem due to A. Aziz on the location of zeros of composite polynomials is given. In the second part some properties of a fixed length continuation of a polynomial are deduced.

In this note we shall indicate a short proof of the main theorem in [1] concerning the zeros of composite polynomials and then apply these results to consider a problem in fixed length continuation of polynomials.

1. The following main theorem was proved in [1] via several lemmas and Grace’s theorem.

Theorem A. If

\[ P(z) = \sum_{j=0}^{n} C(n, j) A_j z^j \quad \text{and} \quad Q(z) = \sum_{j=0}^{m} C(m, j) B_j z^j \]

are two polynomials of degree \( n \) and \( m \) respectively, \( m < n \), such that

\[ C(m,0) A_0 B_m - C(m,1) A_1 B_{m-1} + \cdots + (-1)^m C(m, m) A_m B_0 = 0, \]

then the following holds:

(i) If \( Q(z) \) has all its zeros in the disk \( |z| < r \), then \( P(z) \) has at least one zero in \( |z| < r \).

(ii) If \( P(z) \) has all its zeros in the region \( |z| > r \), then \( Q(z) \) has at least one zero in \( |z| > r \).

We propose the following short proof.

Proof. (i) Relation (1) is invariant under the transformation \( z \to rz \) in \( P(z) \) and \( Q(z) \) so that we may assume \( r = 1 \). Assume by contradiction that all zeros of \( P(z) \) lie in \( |z| > 1 \). By the well known Gauss-Lucas result on the zeros of the derivative of a polynomial, all the zeros of

\[ P_1(z) = D^{(n-m)} \left[ z^m P(1/z) \right] = n(n-1) \cdots (m+1) \sum_{j=0}^{m} C(m, j) A_j z^{m-j} \]

lie in \( |z| \leq 1 \), so that the zeros of

\[ P_2(z) = z^m P_1(1/z) = \sum_{j=0}^{m} C(m, j) A_j z^j \]

lie in \( |z| > 1 \).
The hypothesis (1) means that $P_2(z)$ and $Q(z)$ are apolar, so that by Grace’s theorem [3, Theorem 15.3] the region $|z| > 1$ in the extended complex plane which contains all the zeros of $P_2(z)$ contains at least one zero of $Q(z)$. If the degree of $P_2$ is $m - k$, we shall assume that $P_2$ has $k$ zeros at the point at infinity so that the total number of zeros of $P_2$ is $m$. This is a contradiction to the hypothesis on the zeros of $Q(z)$.

Actually, one can avoid using the extended complex plane by assuming first that $A_m \neq 0$, then applying a continuity argument.

Proof of assertion (ii) is similar to the proof of (i). Since the zeros of $P(z)$ all lie in $|z| > 1$, so do the zeros of $P_2(z)$. The conclusion follows by Grace’s theorem.

2. Let $m$ and $n$ be fixed integers, $1 \leq m \leq n$, and let $p(z) = \sum_{j=0}^{m} C(n, j) A_j z^j$, $A_0 = 1, A_m \neq 0$, be a fixed polynomial. Any polynomial $q(z)$ of the form

$$q(z) = p(z) + a_{m+1} z^{m+1} + \cdots + a_n z^n$$

will be called a continuation of $p(z)$ of length $n$. Denote the family of all such continuations by $\Pi(p, n)$. It is well known [2] that for sufficiently large $n$ the zeros of polynomials in $\Pi$ can be made to lie on an arbitrary Jordan curve which contains the origin in its interior. However, if, as we assumed, $n$ is fixed, there exists a largest disk about the origin free of zeros of all members of $\Pi$. We shall give an estimate of the radius of this disk and formulate a conjecture with regards to its exact value.

For each $q \in \Pi$ let $\mu(q) = \min_{1 \leq k \leq n} |z_k|$, where the $z_k$ are the zeros of $q$ with the convention that $q$ has $(n - \deg q)$ zeros at the point at infinity. We estimate $\rho = \rho(p, n)$ defined by

$$\rho = \sup_{q \in \Pi} \mu(q).$$

A priori it is not clear that $\rho$ is finite. If $A_s$ is the first nonzero coefficient with $s \geq 1$, then it follows by a theorem of Van Vleck [3, Theorem 33.3] that every $q \in \Pi$ has a zero in the disk $|z| < \left[ C(n, s)/|A_s| \right]^{1/s}$. Thus, $0 < \rho < \infty$.

**Theorem.** If $\rho$ is defined by (2), then the inequality

$$\rho \leq \mu \left( \sum_{j=0}^{m} C(m, j) A_j B_j z^j \right)/\mu \left( \sum_{j=0}^{m} C(m, j) B_j z^j \right)$$

holds for any choice of complex numbers $B_0, B_1, \ldots, B_m, B_0 B_m \neq 0$.

For the proof of the theorem we shall need this corollary of Theorem A ([1, Theorem 2] or [3, Theorem 16.1]).

**Theorem B.** If all the zeros of $P(z) = \sum_{j=0}^{n} C(n, j) A_j z^j$ of degree $n$ lie in $|z| \geq r$ and if $Q(z) = \sum_{j=0}^{m} C(m, j) B_j z^j$, $B_0 B_m \neq 0$, $m \leq n$, then every zero $\omega$ of the polynomial $R(z) = \sum_{j=0}^{m} C(m, j) A_j B_j z^j$ has the form $\omega = -\alpha \beta$ where $\beta$ is a zero of $Q$ and $|\alpha| \geq r$.

**Proof of the theorem.** Fix $B_j, j = 0, 1, \ldots, m, B_0 B_m \neq 0$. Let $P^* \in \Pi$ such that $\mu(P^*) \geq \rho - \epsilon$. We may assume that $P^* = P$ of Theorem B and $r = \rho - \epsilon$. By Theorem B, $\mu(R) \geq (\rho - \epsilon)|\beta|$, where $\beta$ is a zero of $Q$. Thus, $\mu(R) \geq (\rho - \epsilon)\mu(Q)$ and letting $\epsilon \to 0, \mu(R) \geq \rho \mu(Q)$. This completes the proof.
One notices that the existence of an extremal polynomial \( P^* \in \Pi \) for which \( \mu(P^*) = \rho \) was not assumed. Indeed, it does not seem simple to answer the question of whether such a polynomial exists for every given \( P \). Choosing \( B_j = 1 \) in (3) and noticing that \( \sum_{j=0}^{m} C(n, j) A_j z^j \in \Pi \) we have

**Corollary.** If \( A_m \neq 0 \), then

\[
\frac{\mu \left( \sum_{j=0}^{m} C(n, j) A_j z^j \right)}{\mu \left( \sum_{j=0}^{m} C(m, j) A_j z^j \right)} < \rho \leq \mu \left( \sum_{j=0}^{m} C(m, j) A_j z^j \right).
\]

Although we have assumed that \( A_0 \neq 0 \), it is easy to see that the last inequalities hold as equalities if \( A_0 = 0 \). Since the \( A_j \) are arbitrary subject to \( A_m \neq 0 \), the corollary shows that, for a given polynomial \( P \) of degree \( m \), there are multiplier sequences independent of the coefficients which increase (decrease) \( \mu(P) \). Their study may be worthwhile but it is much beyond the scope of this note.

We conclude with a

**Conjecture.** There exists an extremal polynomial \( P^* \in \Pi \) for which \( \mu(P^*) = \rho = \inf f(B_0, B_1, \ldots, B_m) \) where \( f(B_0, B_1, \ldots, B_m) \) denotes the right-hand side of inequality (3).

This conjecture is true for \( m = 1 \) and arbitrary \( n \). Indeed, one verifies that for \( p(z) = 1 + az \), \( \rho = n/|a| \), an extremal polynomial is \( (1 + (a/n)z)^n \), \( A_0 = 1 \), \( A_1 = a/n \) so that we have equality in (3).

**References**


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