REMARKS ON A PAPER BY A. AZIZ

ZALMAN RUBINSTEIN

Abstract. The paper consists of two parts. In the first part a short proof of the main theorem due to A. Aziz on the location of zeros of composite polynomials is given. In the second part some properties of a fixed length continuation of a polynomial are deduced.

In this note we shall indicate a short proof of the main theorem in [1] concerning the zeros of composite polynomials and then apply these results to consider a problem in fixed length continuation of polynomials.

1. The following main theorem was proved in [1] via several lemmas and Grace's theorem.

Theorem A. If \( P(z) = \sum_{j=0}^{n} C(n, j)A_jz^j \) and \( Q(z) = \sum_{j=0}^{m} C(m, j)B_jz^j \) are two polynomials of degree \( n \) and \( m \) respectively, \( m \leq n \), such that

\[
C(m, 0)A_0B_m - C(m, 1)A_1B_{m-1} + \cdots + (-1)^m C(m, m)A_mB_0 = 0,
\]

then the following holds:

(i) If \( Q(z) \) has all its zeros in the disk \( |z| < r \), then \( P(z) \) has at least one zero in \( |z| < r \).

(ii) If \( P(z) \) has all its zeros in the region \( |z| > r \), then \( Q(z) \) has at least one zero in \( |z| \geq r \).

We propose the following short proof.

Proof. (i) Relation (1) is invariant under the transformation \( z \rightarrow rz \) in \( P(z) \) and \( Q(z) \) so that we may assume \( r = 1 \). Assume by contradiction that all zeros of \( P(z) \) lie in \( |z| > 1 \). By the well known Gauss-Lucas result on the zeros of the derivative of a polynomial, all the zeros of

\[
P_1(z) = D(n-m)[z^n P(1/z)] = n(n-1) \cdots (m+1) \sum_{j=0}^{m} C(m, j)A_jz^{m-j}
\]

lie in \( |z| < 1 \), so that the zeros of

\[
P_2(z) = z^m P_1(1/z) = \sum_{j=0}^{m} C(m, j)A_jz^j
\]

lie in \( |z| > 1 \).

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The hypothesis (1) means that $P_2(z)$ and $Q(z)$ are apolar, so that by Grace’s theorem [3, Theorem 15.3] the region $|z| > 1$ in the extended complex plane which contains all the zeros of $P_2(z)$ contains at least one zero of $Q(z)$. If the degree of $P_2$ is $m - k$, we shall assume that $P_2$ has $k$ zeros at the point at infinity so that the total number of zeros of $P_2$ is $m$. This is a contradiction to the hypothesis on the zeros of $Q(z)$.

Actually, one can avoid using the extended complex plane by assuming first that $A_m \neq 0$, then applying a continuity argument.

Proof of assertion (ii) is similar to the proof of (i). Since the zeros of $P(z)$ all lie in $|z| > 1$, so do the zeros of $P_2(z)$. The conclusion follows by Grace’s theorem.

2. Let $m$ and $n$ be fixed integers, $1 \leq m \leq n$, and let $p(z) = \sum_{j=0}^{m} C(n, j) A_j z^j$, $A_0 = 1$, $A_m \neq 0$, be a fixed polynomial. Any polynomial $q(z)$ of the form

$$q(z) = p(z) + a_{m+1} z^{m+1} + \cdots + a_n z^n$$

will be called a continuation of $p(z)$ of length $n$. Denote the family of all such continuations by $\Pi(p, n)$. It is well known [2] that for sufficiently large $n$ the zeros of polynomials in $\Pi$ can be made to lie on an arbitrary Jordan curve which contains the origin in its interior. However, if, as we assumed, $n$ is fixed, there exists a largest disk about the origin free of zeros of all members of $\Pi$. We shall give an estimate of the radius of this disk and formulate a conjecture with regards to its exact value.

For each $q \in \Pi$ let $\mu(q) = \min_{1 \leq k \leq n} |z_k|$, where the $z_k$ are the zeros of $q$ with the convention that $q$ has $(n - \deg q)$ zeros at the point at infinity. We estimate

$$\rho = \rho(p, n)$$

defined by

$$(2) \quad \rho = \sup_{q \in \Pi} \mu(q).$$

A priori it is not clear that $\rho$ is finite. If $A_s$ is the first nonzero coefficient with $s > 1$, then it follows by a theorem of Van Vleck [3, Theorem 33.3] that every $q \in \Pi$ has a zero in the disk $|z| < \left[ \frac{C(n, s)}{|A_s|} \right]^{1/s}$. Thus, $0 < \rho < \infty$.

**Theorem.** If $\rho$ is defined by (2), then the inequality

$$(3) \quad \rho \leq \mu \left( \sum_{j=0}^{m} C(m, j) A_j B_j z^j \right) / \mu \left( \sum_{j=0}^{m} C(m, j) B_j z^j \right)$$

holds for any choice of complex numbers $B_0, B_1, \ldots, B_m, B_0 B_m \neq 0$.

For the proof of the theorem we shall need this corollary of Theorem A ([1, Theorem 2] or [3, Theorem 16.1]).

**Theorem B.** If all the zeros of $P(z) = \sum_{j=0}^{n} C(n, j) A_j z^j$ of degree $n$ lie in $|z| \geq r$ and if $Q(z) = \sum_{j=0}^{m} C(m, j) B_j z^j$, $B_0 B_m \neq 0$, $m \leq n$, then every zero $\omega$ of the polynomial $R(z) = \sum_{j=0}^{m} C(m, j) A_j B_j z^j$ has the form $\omega = -\alpha \beta$ where $\beta$ is a zero of $Q$ and $|\alpha| \geq r$.

**Proof of the theorem.** Fix $B_j, j = 0, 1, \ldots, m, B_0 B_m \neq 0$. Let $P^* \in \Pi$ such that $\mu(P^*) \geq \rho - \epsilon$. We may assume that $P^* = P$ of Theorem B and $r = \rho - \epsilon$. By Theorem B, $\mu(R) \geq (\rho - \epsilon) |\beta|$, where $\beta$ is a zero of $Q$. Thus, $\mu(R) \geq (\rho - \epsilon) \mu(Q)$ and letting $\epsilon \to 0$, $\mu(R) \geq \rho \mu(Q)$. This completes the proof.
One notices that the existence of an extremal polynomial $P^* \in \Pi$ for which
\[ \mu(P^*) = \rho \]
was not assumed. Indeed, it does not seem simple to answer the question of whether such a polynomial exists for every given $P$. Choosing $B_j = 1$ in (3) and noticing that $\sum_{j=0}^{m} C(n, j) A_j z^j \in \Pi$ we have

**Corollary.** If $A_m \neq 0$, then
\[ \mu \left( \sum_{j=0}^{m} C(n, j) A_j z^j \right) \leq \rho \leq \mu \left( \sum_{j=0}^{m} C(m, j) A_j z^j \right). \]

Although we have assumed that $A_0 \neq 0$, it is easy to see that the last inequalities hold as equalities if $A_0 = 0$. Since the $A_j$ are arbitrary subject to $A_m \neq 0$, the corollary shows that, for a given polynomial $P$ of degree $m$, there are multiplier sequences independent of the coefficients which increase (decrease) $\mu(P)$. Their study may be worthwhile but it is much beyond the scope of this note.

We conclude with a

**Conjecture.** There exists an extremal polynomial $P^* \in \Pi$ for which $\mu(P^*) = \rho = \inf f(B_0, B_1, \ldots, B_m)$ where $f(B_0, B_1, \ldots, B_m)$ denotes the right-hand side of inequality (3).

This conjecture is true for $m = 1$ and arbitrary $n$. Indeed, one verifies that for $p(z) = 1 + az$, $\rho = n/|a|$, an extremal polynomial is $(1 + (a/n)z)^n$, $A_0 = 1$, $A_1 = a/n$ so that we have equality in (3).

**References**


**Department of Mathematics, University of Haifa, Mount Carmel, Haifa, Israel**

**Current address:** Department of Mathematics, University of Colorado, Boulder, Colorado 80309