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REMARKS ON A PAPER BY A. AZIZ

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ABSTRACT. The paper consists of two parts. In the first part a short proof of the main theorem due to A. Aziz on the location of zeros of composite polynomials is given. In the second part some properties of a fixed length continuation of a polynomial are deduced.

In this note we shall indicate a short proof of the main theorem in **[1]** concerning the zeros of composite polynomials and then apply these results to consider a problem in fixed length continuation of polynomials.

1. The following main theorem was proved in **[1]** via several lemmas and Grace's theorem.

THEOREM A. *If* $P(z) = \sum_{j=0}^{n} C(n, j) A_j z^j$ *and* $Q(z) = \sum_{j=0}^{m} C(m, j) B_j z^j$ *are two polynomials of degree n and m respectively,* $m \leqslant n$, *such that*

$$(1) \qquad C(m,0)A_0B_m - C(m,1)A_1B_{m-1} + \cdots + (-1)^m C(m, m) A_m B_0 = 0,$$

then the following holds:

(i) *If* $Q(z)$ *has all its zeros in the disk* $|z| \leqslant r$, *then* $P(z)$ *has at least one zero in* $|z| \leqslant r$.

(ii) *If* $P(z)$ *has all its zeros in the region* $|z| \geqslant r$, *then* $Q(z)$ *has at least one zero in* $|z| \geqslant r$.

We propose the following short

PROOF. (i) Relation (1) is invariant under the transformation $z \to rz$ in $P(z)$ and $Q(z)$ so that we may assume $r = 1$. Assume by contradiction that all zeros of $P(z)$ lie in $|z| > 1$. By the well known Gauss-Lucas result on the zeros of the derivative of a polynomial, all the zeros of

$$P_1(z) = D^{(n-m)}[z^n P(1/z)] = n(n-1)\cdots(m+1) \sum_{j=0}^{m} C(m, j) A_j z^{m-j}$$

lie in $|z| \leqslant 1$, so that the zeros of

$$P_2(z) = z^m P_1(1/z) = \sum_{j=0}^{m} C(m, j) A_j z^j$$

lie in $|z| > 1$.

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The hypothesis (1) means that $P_2(z)$ and $Q(z)$ are apolar, so that by Grace's theorem [**3**, Theorem 15.3] the region $|z| > 1$ in the extended complex plane which contains all the zeros of $P_2(z)$ contains at least one zero of $Q(z)$. If the degree of P_2 is $m - k$, we shall assume that P_2 has k zeros at the point at infinity so that the total number of zeros of P_2 is m. This is a contradiction to the hypothesis on the zeros of $Q(z)$.

Actually, one can avoid using the extended complex plane by assuming first that $A_m \neq 0$, then applying a continuity argument.

Proof of assertion (ii) is similar to the proof of (i). Since the zeros of $P(z)$ all lie in $|z| \geqslant 1$, so do the zeros of $P_2(z)$. The conclusion follows by Grace's theorem.

2. Let m and n be fixed integers, $1 \leqslant m \leqslant n$, and let $p(z) = \sum_{j=0}^{m} C(n, j) A_j z^j$, $A_0 = 1$, $A_m \neq 0$, be a fixed polynomial. Any polynomial $q(z)$ of the form

$$q(z) = p(z) + a_{m+1} z^{m+1} + \cdots + a_n z^n$$

will be called a continuation of $p(z)$ of length n. Denote the family of all such continuations by $\Pi(p, n)$. It is well known [**2**] that for sufficiently large n the zeros of polynomials in Π can be made to lie on an arbitrary Jordan curve which contains the origin in its interior. However, if, as we assumed, n is fixed, there exists a largest disk about the origin free of zeros of all members of Π. We shall give an estimate of the radius of this disk and formulate a conjecture with regards to its exact value.

For each $q \in \Pi$ let $\mu(q) = \min_{1 \leqslant k \leqslant n} |z_k|$, where the z_k are the zeros of q with the convention that q has $(n - \deg q)$ zeros at the point at infinity. We estimate $\rho = \rho(p, n)$ defined by

$$\rho = \sup_{q \in \Pi} \mu(q). \tag{2}$$

A priori it is not clear that ρ is finite. If A_s is the first nonzero coefficient with $s \geqslant 1$, then it follows by a theorem of Van Vleck [**3**, Theorem 33.3] that every $q \in \Pi$ has a zero in the disk $|z| \leqslant [C(n, s)/|A_s|]^{1/s}$. Thus, $0 < \rho < \infty$.

THEOREM. *If ρ is defined by* (2), *then the inequality*

$$\rho \leqslant \mu\left(\sum_{j=0}^{m} C(m, j) A_j B_j z^j \right) \Big/ \mu\left(\sum_{j=0}^{m} C(m, j) B_j z^j \right) \tag{3}$$

holds for any choice of complex numbers $B_0, B_1, \ldots, B_m$, $B_0 B_m \neq 0$.

For the proof of the theorem we shall need this corollary of Theorem A ([**1**, Theorem 2] or [**3**, Theorem 16.1]).

THEOREM B. *If all the zeros of $P(z) = \sum_{j=0}^{n} C(n, j) A_j z^j$ of degree n lie in $|z| \geqslant r$ and if $Q(z) = \sum_{j=0}^{m} C(m, j) B_j z^j$, $B_0 B_m \neq 0$, $m \leqslant n$, then every zero ω of the polynomial $R(z) = \sum_{j=0}^{m} C(m, j) A_j B_j z^j$ has the form $\omega = -\alpha\beta$ where β is a zero of Q and $|\alpha| \geqslant r$.*

PROOF OF THE THEOREM. Fix B_j, $j = 0, 1, \ldots, m$, $B_0 B_m \neq 0$. Let $P^* \in \Pi$ such that $\mu(P^*) \geqslant \rho - \varepsilon$. We may assume that $P^* = P$ of Theorem B and $r = \rho - \varepsilon$. By Theorem B, $\mu(R) \geqslant (\rho - \varepsilon)|\beta|$, where β is a zero of Q. Thus, $\mu(R) \geqslant (\rho - \varepsilon)\mu(Q)$ and letting $\varepsilon \to 0$, $\mu(R) \geqslant \rho\mu(Q)$. This completes the proof.

One notices that the existence of an extremal polynomial $P^* \in \Pi$ for which $\mu(P^*) = \rho$ was not assumed. Indeed, it does not seem simple to answer the question of whether such a polynomial exists for every given P. Choosing $B_j = 1$ in (3) and noticing that $\sum_{j=0}^{m} C(n, j) A_j z^j \in \Pi$ we have

COROLLARY. *If* $A_m \neq 0$, *then*

$$\mu\left(\sum_{j=0}^{m} C(n, j) A_j z^j\right) \leqslant \rho \leqslant \mu\left(\sum_{j=0}^{m} C(m, j) A_j z^j\right).$$

Although we have assumed that $A_0 \neq 0$, it is easy to see that the last inequalities hold as equalities if $A_0 = 0$. Since the A_j are arbitrary subject to $A_m \neq 0$, the corollary shows that, for a given polynomial P of degree m, there are multiplier sequences idependent of the coefficients which increase (decrease) $\mu(P)$. Their study may be worthwhile but it is much beyond the scope of this note.

We conclude with a

Conjecture. There exists an extremal polynomial $P^* \in \Pi$ for which $\mu(P^*) = \rho = \inf f(B_0, B_1, \ldots, B_m)$ where $f(B_0, B_1, \ldots, B_m)$ denotes the right-hand side of inequality (3).

This conjecture is true for $m = 1$ and arbitrary n. Indeed, one verifies that for $p(z) = 1 + az$, $\rho = n/|a|$, an extremal polynomial is $(1 + (a/n)z)^n$, $A_0 = 1$, $A_1 = a/n$ so that we have equality in (3).

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