ON THE MULTIPLICITIES OF THE POWERS OF A BANACH SPACE OPERATOR

DOMINGO A. HERRERO

Abstract. The multiplicities of the powers of a bounded linear operator $T$, acting on a complex separable infinite-dimensional Banach space $X$, satisfy the inequalities

\[(**)
\mu(T^n) \leq \mu(T^{hn}) \leq h\mu(T^n)
\]

for all $h, n \geq 1$.

Nothing else can be said, in general, because simple examples show that for each sequence $\{\mu_n\}_{n=1}^\infty$, satisfying the inequalities $(**)$, there exists $T$ acting on $X$ such that $\mu(T^n) = \mu_n$ for all $n \geq 1$.

Let $\mathcal{L}(X)$ denote the algebra of all (bounded linear) operators acting on a complex separable infinite-dimensional Banach space $X$. The multiplicity of $T$ in $\mathcal{L}(X)$ is the cardinal number defined by

\[p(T) = \min \{c(T) : \text{cyclic}(T) = \mathcal{V}\{T^n : y \in \text{cyclic}(X), n = 0, 1, 2, \ldots\}\},\]

where $\mathcal{V}X$ denotes the closed linear span of the vectors in $X$. It is immediate from the definition that

\[(1) \quad \mu(T^n) \leq \mu(T^{hn}) \leq h\mu(T^n) \quad \text{for all } h, n \geq 1.\]

Thus, in particular, if $T^n$ is cyclic (i.e., $\mu(T^n) = 1$) for some $n \geq 1$, then $T^m$ is also cyclic for all $m|n$ (where $m|n$ indicates that $m$ divides $n$).

This is the only possible general result relating the multiplicities of the different powers of a given operator. Indeed, we have the following result.

**Theorem 1.** Let $X$ be a complex separable infinite-dimensional Banach space. Given a sequence $M = \{\mu_n\}_{n=1}^\infty$ of natural numbers, satisfying the inequalities

\[(2) \quad \mu_n \leq \mu_{hn} \leq h\mu_n \quad \text{for all } h, n \geq 1,\]

there exists a nuclear operator $T(M)$ in $\mathcal{L}(X)$ such that

\[\mu(T(M)^n) = \mu_n \quad \text{for all } n = 1, 2, \ldots.\]

For the Hilbert space case, we have

**Theorem 2.** Let $H$ be a complex separable infinite-dimensional Hilbert space. Given a sequence $M$ satisfying the conditions of Theorem 1, there exists a normal operator $N(M)$ in $\mathcal{L}(H)$ such that

\[\mu(N(M)^n) = \mu_n \quad \text{for all } n = 1, 2, \ldots, \quad \text{and} \quad \sigma(N(M)) = \{\lambda : |\lambda| \leq 1\},\]

where $\sigma(N(M))$ denotes the spectrum of $N(M)$.

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Let \( \{e_i\}_{i=1}^{n} \) be the canonical orthonormal basis of \( \mathbb{C}^n \) and let \( e(t) = \exp\{2\pi i t\} \) (\( t \) a real number). If \( \alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} = \mu_n \), then we write
\[
A = A(\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}) = \bigoplus_{j=0}^{n-1} e\left(\frac{j}{n}\right)^{(\alpha_j)}
\]
to indicate the diagonal (and therefore normal) operator defined by
\[
Ae_i = \begin{cases} 
  e_i, & 1 \leq i \leq \alpha_0, \\
  e(j/n)e_i, & \alpha_0 + \alpha_1 + \cdots + \alpha_{j-1} < i \leq \alpha_0 + \alpha_1 + \cdots + \alpha_j \\
  0, & (j = 1, 2, \ldots, n-1).
\end{cases}
\]

Observe that \( A^n \) is the identity operator, and therefore
\[
\mu(A^n) = \dim \mathbb{C}^n = \mu_n.
\]

It is clear that every operator satisfying this condition must also satisfy
\[
\mu(A^m) \geq \text{(smallest integer greater than or equal to } m\mu/n \text{)}
\]
\[
= \left[\left(m\mu + n - 1\right)/n\right] = \left[\left(m\mu - 1\right)/n\right] + 1
\]
for each \( m|n \), where \([t]\) denotes the integral part of the real number \( t \). (To see this, use (1).)

Observe that, for each \( k \geq 1 \),
\[
\mu(A^k) = n \sum_{i=0}^{n-1} e\left(\frac{j}{n}\right)^{(\alpha_i)} = \max_{0 \leq j < n} \left[\alpha_j: kj \equiv t \pmod{n}\right]
\]
(with the convention that \( \sum\{\alpha_j: kj \equiv t \pmod{n}\} = 0 \) if \( kj \not\equiv t \pmod{n} \) for all \( j = 0, 1, 2, \ldots, n-1 \)).

The key result is Lemma 3 below, which says that, for a clever choice of \( \alpha_0, \alpha_1, \ldots, \alpha_{n-1} \), \( \mu(A^k) \) does not exceed \( \mu(B^k) \) for any operator \( B \) such that \( \mu(B^n) = \mu_n \) (for all \( k = 1, 2, \ldots \)).

**Lemma 3.** Let \( n, \mu_n \geq 1 \) and let
\[
s_n = \left[\left(\mu_n - 1\right)/n\right] + 1 \quad \text{and} \quad \alpha_n = \mu_n - n(s_n - 1),
\]
if
\[
A_n = \bigoplus_{j=0}^{a_n-1} e\left(\frac{j}{n}\right)^{(s_n)} \bigoplus_{j=a_n}^{n-1} e\left(\frac{j}{n}\right)^{(s_n-1)}
\]
then \( \mu(A_n^m) = \mu_n \) and
\[
\mu(A_n^k) = \mu(A_n^{k,n}) = \left[\left((k, n)\mu_n - 1\right)/n\right] + 1
\]
\[
= \min\{\mu(B^k): B \in \mathcal{L}(\mathcal{X}), \mu(B^n) = \mu_n\}
\]
for all \( k = 1, 2, \ldots \), where \( (k, n) = \text{G.C.D.}\{k, n\} \) and \( \mathcal{X} \) is an arbitrary complex separable Banach space with \( \dim \mathcal{X} \geq \mu_n \).
Proof. It follows from (3) and (5) that
\[ \mu(A^n) = \mu(A^n) = \mu_n \quad (h = 1, 2, \ldots) \]
and
\[ \mu(A_n) = \max\{s_n, s_n - 1\} = s_n. \]

Let \( Z_n = Z/nZ \). Observe that if \((k, n) = 1\), then the application “multiplication by \( k \)” is an automorphism of the ring \( Z_n \), whence we easily deduce (by using (5)) that \( \mu(A^k_n) = \mu(A_n) = s_n \). More generally, if \((k, n) = m, k = ma, n = mb\), then “multiplication by \( k \)” = (“multiplication by \( a \”)° (“multiplication by \( m \”\); the image of “multiplication by \( m \)” is \( mZ_n = Z/(n/m) = Z_h \), and “multiplication by \( a \)” is an automorphism of the subring \( mZ_n \) because \((a, b) = 1\). It follows from these observations, along with (4) and (5), that
\[ \mu(A^k_n) = \mu(A_n) = \left\lfloor \frac{(m\mu_n - 1)}{n} \right\rfloor + 1 \quad \text{for all } k = 1, 2, \ldots. \]

Thus, in order to complete the proof, we only have to show that if \( 1 < m < n \) and \( m|n \), then \( \mu(A^m_n) = \left\lfloor \frac{(m\mu_n - 1)}{n} \right\rfloor + 1 \). By using (5), we have
\[
\begin{align*}
\mu(A^m_n) &= \max_{0 \leq t < n} \{ s_n \cdot c\{ j : jm \equiv t \pmod{n}, 0 \leq j < a_n \} \\
&\quad + (s_n - 1)(c\{ j : jm \equiv t \pmod{n}, a_n \leq j < n \}) \\
&= \max_{0 \leq t < n} \{ s_n \cdot c\{ j : jm \equiv t \pmod{n} \} - c\{ j : jm \equiv t \pmod{n}, a_n \leq j < n \} \}
\end{align*}
\]
\[
= s_n \cdot c\{ j : jm \equiv 0 \pmod{n} \} - c\{ j : jm \equiv 0 \pmod{n}, a_n \leq j < n \}
= ms_n - c\{ l : a_n \leq l(n/m) < n \} \quad \text{(using the fact that } \alpha_0 = s_n, \alpha_1 = s_n, \alpha_2 = s_n - 1, \ldots, \alpha_{a_n - 1} = s_n - 1 \text{ is a nonincreasing sequence)}
\]
\[
= ms_n - c\{ l : (m/n)a_n \leq l < m \} = ms_n - \left\lfloor m - (ma_n/n) \right\rfloor
\]
\[
= ms_n - \left\lfloor ms_n - (m\mu_n/n) \right\rfloor
= m\left(\mu_n - 1\right)/n + m - \left[ m\left(\mu_n - 1\right)/n \right] + m - (m\mu_n/n) \quad \text{(using (6)).}
\]

Let \( \mu_n = hn + g \), where \( h \geq 0, 0 \leq g < n \); then a straightforward computation shows that
\[
\mu(A^m_n) = \begin{cases} 
mh & \text{if } g = 0, \\
mh + m - \left[ m - (mg/n) \right] & \text{if } 1 \leq g < n.
\end{cases}
\]

On the other hand,
\[
\left\lfloor (m\mu_n - 1)/n \right\rfloor + 1 = \begin{cases} 
mh & \text{if } g = 0, \\
\left[mh + ((mg - 1)/n)\right] + 1 & \text{if } 1 \leq g < n.
\end{cases}
\]

Thus, \( \mu(A^m_n) = \left\lfloor (m\mu_n - 1)/n \right\rfloor + 1 = mh \) for the case when \( g = 0 \). If \( ln/m < g \leq (l + 1)(n/m) \), then
\[
\mu(A^m_n) = \left\lfloor (m\mu_n - 1)/n \right\rfloor = mh + l + 1, \quad l = 0, 1, 2, \ldots, m - 1,
\]
whence we conclude that \( \mu(A^m_n) = \left\lfloor (m\mu_n - 1)/n \right\rfloor + 1 \) for all \( m|n, 1 < m < n \).

The proof of Lemma 3 is now complete. □
Proof of Theorem 1. Suppose that $\mathcal{H}$ is a Hilbert space with orthonormal basis $\{e_i\}_{i=1}^\infty$. Define

$$A_1 = 1^{(\mu_1)} \text{ on } V\{e_1, e_2, \ldots, e_{\mu_1}\},$$
$$A_2 \text{ on } V\{e_{\mu_1+1}, e_{\mu_1+2}, \ldots, e_{\mu_1+\mu_2}\},$$
$$\vdots$$
$$A_n \text{ on } V\{e_{\mu_1+\mu_2+\ldots+\mu_{n-1}+1}, e_{\mu_1+\mu_2+\ldots+\mu_{n-1}+2}, \ldots, e_{\mu_1+\mu_2+\ldots+\mu_n}\},$$

exactly as in Lemma 3.

Let $\{r_n\}_{n=1}^\infty$ be any strictly decreasing sequence of positive reals with $r_1 = 1$. Now we define

$$T(M) = \bigoplus_{n=1}^\infty r_n A_n \in \mathcal{L}(\mathcal{H}).$$

It is easy to check that $T(M)$ is normal, $\|T(M)\| = 1$,

$$\sigma(T(M)) \subset \left( \bigcup_{n=1}^\infty \left\{ r_n e\left(\frac{j}{n}\right) : j = 0, 1, 2, \ldots, n-1 \right\} \right)^\sim,$$

and

$$\mu(T(M)^k) = \sup_n \mu(A_n^k) \quad (k = 1, 2, \ldots).$$

By using (2) and Lemma 3, we deduce that

$$\mu(A_n^k) = \mu(A_n^{(k,n)}) = \left[ ((k, n) \mu_n - 1)/n \right] + 1 \leq \mu_k = \mu(A_k^k)$$

for all $n = 1, 2, \ldots$, and therefore

$$\mu(T(M)^k) = \mu_k \quad \text{for all } k = 1, 2, \ldots.$$

Furthermore, if $r_n \downarrow 0$ fast enough, then $T(M)$ is a nuclear operator. (It suffices to take $r_n = (n + \mu_n)^{-\frac{1}{2}}, n = 1, 2, \ldots$.)

This proves Theorem 1 for the case when $\mathcal{H}$ is a Hilbert space. If $\mathcal{H}$ is not a Hilbert space, then it is enough to repeat the above construction with the orthonormal basis replaced by a normalized Markushevich basis (see [2]; the details are left to the reader). $\Box$

Proof of Theorem 2. We begin by constructing a nuclear normal operator $T(M)$ exactly as in the previous proof.

Let $L$ be a diagonal normal operator defined by $Lf_{ij} = t_i e(v_j)f_{ij}$ with respect to an orthonormal basis $\{f_{ij}\}_{i,j=1}^\infty$, where $\{t_i\}_{i=1}^\infty$ is a denumerable dense subset of (distinct points of) $(0,1) \setminus \{r_n\}_{n=1}^\infty$ and $\{e(v_j)\}_{j=1}^\infty$ is a denumerable dense subset of the unit circle such that $v_j$ and $v_j/v_h$ are irrational for all $j$ and, respectively, for all $h \neq j$. Then $L$ is a normal operator, the set of all eigenvalues of $L^k$ is disjoint from the set of all eigenvalues of $T(M)^k$ for each $k = 1, 2, \ldots$, and it straightforward to check that

$$\mu\left(\{T(M) \oplus L\}^k\right) = \max\{\mu(T(M)^k), \mu(L^k)\}$$

$$= \max\{\mu_k, 1\} = \mu_k \quad \text{for all } k = 1, 2, \ldots.$$
(see [1]), and

\[ \sigma(T(M) \oplus L) = \sigma(L) = \{ \lambda : |\lambda| \leq 1 \}. \]

Thus, the normal operator \( N(M) = T(M) \oplus L \) satisfies all our requirements.

REFERENCES


DEPARTMENT OF MATHEMATICS, ARIZONA STATE UNIVERSITY, TEMPE, ARIZONA 85287