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ON THE MULTIPLICITIES OF THE POWERS OF A BANACH SPACE OPERATOR

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ABSTRACT. The multiplicities of the powers of a bounded linear operator T, acting on a complex separable infinite-dimensional Banach space $\mathscr{X}$, satisfy the inequalities

$$(**)\qquad \mu(T^n) \leqslant \mu(T^{hn}) \leqslant h\mu(T^n) \quad \text{for all } h, n \geqslant 1.$$

Nothing else can be said, in general, because simple examples show that for each sequence $\{\mu_n\}_{n=1}^{\infty}$, satisfying the inequalities $(**)$, there exists T acting on $\mathscr{X}$ such that $\mu(T^n) = \mu_n$ for all $n \geqslant 1$.

Let $\mathscr{L}(\mathscr{X})$ denote the algebra of all (bounded linear) operators acting on a complex separable infinite-dimensional Banach space $\mathscr{X}$. The *multiplicity* of T in $\mathscr{L}(\mathscr{X})$ is the cardinal number defined by

$$\mu(T) = \min_{\Gamma \subset \mathscr{X}} \{\mathbf{c}(\Gamma) : \mathscr{X} = \bigvee\{T^n y : y \in \Gamma, n = 0, 1, 2, \ldots\}\},$$

where $\bigvee\mathscr{R}$ denotes the closed linear span of the vectors in $\mathscr{R}$. It is immediate from the definition that

$$(1)\qquad \mu(T^n) \leqslant \mu(T^{hn}) \leqslant h\mu(T^n) \quad \text{for all } h, n \geqslant 1.$$

Thus, in particular, if T^n is cyclic (i.e., $\mu(T^n) = 1$) for some $n \geqslant 1$, then T^m is also cyclic for all $m|n$ (where $m|n$ indicates that m divides n).

This is the only possible general result relating the multiplicities of the different powers of a given operator. Indeed, we have the following result.

THEOREM 1. *Let $\mathscr{X}$ be a complex separable infinite-dimensional Banach space. Given a sequence $M = \{\mu_n\}_{n=1}^{\infty}$ of natural numbers, satisfying the inequalities*

$$(2)\qquad \mu_n \leqslant \mu_{hn} \leqslant h\mu_n \quad \text{for all } h, n \geqslant 1,$$

there exists a nuclear operator $T(M)$ in $\mathscr{L}(\mathscr{X})$ such that

$$\mu(T(M)^n) = \mu_n \quad \text{for all } n = 1, 2, \ldots.$$

For the Hilbert space case, we have

THEOREM 2. *Let $\mathscr{H}$ be a complex separable infinite-dimensional Hilbert space. Given a sequence M satisfying the conditions of Theorem 1, there exists a normal operator $N(M)$ in $\mathscr{L}(\mathscr{H})$ such that*

$$\mu(N(M)^n) = \mu_n \quad \text{for all } n = 1, 2, \ldots, \quad \text{and} \quad \sigma(N(M)) = \{\lambda : |\lambda| \leqslant 1\},$$

where $\sigma(N(M))$ denotes the spectrum of $N(M)$.

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Let $\{e_i\}_{i=1}^{\mu_n}$ be the canonical orthonormal basis of $\mathbf{C}^{\mu_n}$ and let $e(t) = \exp\{2\pi i t\}$ (t a real number). If $\alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} = \mu_n$, then we write

$$A = A(\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}) = \bigoplus_{j=0}^{n-1} e\left(\frac{j}{n}\right)^{(\alpha_j)}$$

to indicate the diagonal (and therefore *normal*) operator defined by

$$Ae_i = \begin{cases} e_i, & 1 \leqslant i \leqslant \alpha_0, \\ e(j/n)e_i, & \alpha_0 + \alpha_1 + \cdots + \alpha_{j-1} < i \leqslant \alpha_0 + \alpha_1 + \cdots + \alpha_j \\ & (j = 1, 2, \ldots, n-1). \end{cases}$$

Observe that A^n is the identity operator, and therefore

(3) $$\mu(A^n) = \dim \mathbf{C}^{\mu_n} = \mu_n.$$

It is clear that every operator satisfying this condition must also satisfy

(4) $$\begin{aligned} \mu(A^m) &\geqslant \{\text{smallest integer greater than or equal to } m\mu_n/n\} \\ &= [(m\mu_n + n - 1)/n] = [(m\mu_n - 1)/n] + 1 \end{aligned}$$

for each $m|n$, where $[t]$ denotes the integral part of the real number t. (To see this, use (1).)

Observe that, for each $k \geqslant 1$,

(5) $$\mu(A^k) = \mu\left(\bigoplus_{j=0}^{n-1} e\left(\frac{kj}{n}\right)^{(\alpha_j)}\right) = \max_{0 \leqslant t < n} \Sigma\{\alpha_j : kj \equiv t \ (\mathrm{mod}\ n)\}$$

(with the convention that $\Sigma\{\alpha_j: kj \equiv t \ (\mathrm{mod}\ n)\} = 0$ if $kj \not\equiv t \ (\mathrm{mod}\ n)$ for all $j = 0, 1, 2, \ldots, n-1$).

The key result is Lemma 3 below, which says that, for a clever choice of $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$, $\mu(A^k)$ does not exceed $\mu(B^k)$ for any operator B such that $\mu(B^n) = \mu_n$ (for all $k = 1, 2, \ldots$).

LEMMA 3. *Let $n, \mu_n \geqslant 1$ and let*

(6) $$s_n = [(\mu_n - 1)/n] + 1 \quad \text{and} \quad a_n = \mu_n - n(s_n - 1).$$

If

$$A_n = \left\{\bigoplus_{j=0}^{a_n - 1} e\left(\frac{j}{n}\right)^{(s_n)}\right\} \oplus \left\{\bigoplus_{j=a_n}^{n-1} e\left(\frac{j}{n}\right)^{(s_n - 1)}\right\},$$

then $\mu(A_n^n) = \mu_n$ and

$$\begin{aligned} \mu(A_n^k) = \mu\left(A_n^{(k,n)}\right) &= [((k,n)\mu_n - 1)/n] + 1 \\ &= \min\{\mu(B^k): B \in \mathscr{L}(\mathscr{X}), \mu(B^n) = \mu_n\} \end{aligned}$$

for all $k = 1, 2, \ldots$, where $(k, n) =$ G.C.D.$\{k, n\}$ and $\mathscr{X}$ is an arbitrary complex separable Banach space with $\dim \mathscr{X} \geqslant \mu_n$.

PROOF. It follows from (3) and (5) that

$$\mu(A_n^n) = \mu(A_n^{hn}) = \mu_n \qquad (h = 1, 2, \ldots)$$

and

$$\mu(A_n) = \max\{s_n, s_n - 1\} = s_n.$$

Let $\mathbf{Z}_n = \mathbf{Z}/n\mathbf{Z}$. Observe that if $(k, n) = 1$, then the application "multiplication by k" is an automorphism of the ring $\mathbf{Z}_n$, whence we easily deduce (by using (5)) that $\mu(A_n^k) = \mu(A_n) = s_n$. More generally, if $(k, n) = m$, $k = ma$, $n = mb$, then "multiplication by k" = ("multiplication by a")$\circ$("multiplication by m"); the image of "multiplication by m" is $m\mathbf{Z}_n \simeq \mathbf{Z}_{(n/m)} = \mathbf{Z}_b$, and "multiplication by a" is an automorphism of the subring $m\mathbf{Z}_n$ because $(a, b) = 1$. It follows from these observations, along with (4) and (5), that

$$\mu(A_n^k) = \mu(A_n^m) \geqslant [(m\mu_n - 1)/n] + 1 \quad \text{for all } k = 1, 2, \ldots.$$

Thus, in order to complete the proof, we only have to show that if $1 < m < n$ and $m | n$, then $\mu(A_n^m) = [(m\mu_n - 1)/n] + 1$. By using (5), we have

$$\begin{aligned}
\mu(A_n^m) &= \max_{0 \leqslant t < n} \{s_n(\mathbf{c}\{j : jm \equiv t \ (\mathrm{mod}\ n), 0 \leqslant j < a_n\} \\
&\qquad + (s_n - 1)(\mathbf{c}\{j : jm \equiv t \ (\mathrm{mod}\ n), a_n \leqslant j < n\}) \\
&= \max_{0 \leqslant t < n} \{s_n \cdot \mathbf{c}\{j : jm \equiv t \ (\mathrm{mod}\ n)\} - \mathbf{c}\{j : jm \equiv t \ (\mathrm{mod}\ n), a_n \leqslant j < n\}\} \\
&= s_n \cdot \mathbf{c}\{j : jm \equiv 0 \ (\mathrm{mod}\ n)\} - \mathbf{c}\{j : jm \equiv 0 \ (\mathrm{mod}\ n), a_n \leqslant j < n\} \\
&= ms_n - \mathbf{c}\{l : a_n \leqslant l(n/m) < n\} \quad (\text{using the fact that } \alpha_0 = s_n, \alpha_1 = s_n, \\
&\qquad \alpha_1 = s_n, \ldots, \alpha_{a_n - 1} = s_n, \alpha_{a_n} = s_n - 1, \ldots, \\
&\qquad \alpha_{n-1} = s_n - 1 \text{ is a nonincreasing sequence}) \\
&= ms_n - \mathbf{c}\{l : (m/n)a_n \leqslant l < m\} = ms_n - [m - (ma_n/n)] \\
&= ms_n - [ms_n - (m\mu_n/n)] \\
&= m[(\mu_n - 1)/n] + m - [m[(\mu_n - 1)/n] + m - (m\mu_n/n)] \quad (\text{using (6)}).
\end{aligned}$$

Let $\mu_n = hn + g$, where $h \geqslant 0$, $0 \leqslant g < n$; then a straightforward computation shows that

$$\mu(A_n^m) = \begin{cases} mh & \text{if } g = 0, \\ mh + m - [m - (mg/n)] & \text{if } 1 \leqslant g < n. \end{cases} \tag{7}$$

On the other hand,

$$[(m\mu_n - 1)/n] + 1 = \begin{cases} mh & \text{if } g = 0, \\ mh + [(mg - 1)/n] + 1 & \text{if } 1 \leqslant g < n. \end{cases}$$

Thus, $\mu(A_n^m) = [(m\mu_n - 1)/n] + 1 = mh$ for the case when $g = 0$. If $ln/m < g \leqslant (l + 1)(n/m)$, then

$$\mu(A_n^m) = [(m\mu_n - 1)/n] = mh + l + 1, \qquad l = 0, 1, 2, \ldots, m - 1,$$

whence we conclude that $\mu(A_n^m) = [(m\mu_n - 1)/n] + 1$ for all $m | n$, $1 < m < n$.

The proof of Lemma 3 is now complete. □

PROOF OF THEOREM 1. Suppose that $\mathscr{H}$ is a Hilbert space with orthonormal basis $\{e_i\}_{i=1}^{\infty}$. Define

$$\begin{aligned} &A_1 = 1^{(\mu_1)} \quad \text{on} \bigvee\{e_1, e_2, \dots, e_{\mu_1}\},\\ &A_2 \quad \text{on} \bigvee\{e_{\mu_1+1}, e_{\mu_1+2}, \dots, e_{\mu_1+\mu_2}\},\\ &\qquad \vdots\\ &A_n \quad \text{on} \bigvee\{e_{\mu_1+\mu_2+\dots+\mu_{n-1}+1}, e_{\mu_1+\mu_2+\dots+\mu_{n-1}+2}, \dots, e_{\mu_1+\mu_2+\dots+\mu_n}\},\\ &\qquad \vdots \end{aligned}$$

exactly as in Lemma 3.

Let $\{r_n\}_{n=1}^{\infty}$ be any strictly decreasing sequence of positive reals with $r_1 = 1$. Now we define

$$T(M) = \bigoplus_{n=1}^{\infty} r_n A_n \in \mathscr{L}(\mathscr{H}).$$

It is easy to check that $T(M)$ is normal, $\|T(M)\| = 1$,

$$\sigma(T(M)) \subset \left(\bigcup_{n=1}^{\infty} \left\{ r_n e\left(\frac{j}{n}\right) : j = 0, 1, 2, \dots, n-1 \right\} \right)^{-},$$

and

$$\mu\left(T(M)^k\right) = \sup_n \mu\left(A_n^k\right) \qquad (k = 1, 2, \dots).$$

By using (2) and Lemma 3, we deduce that

$$\mu\left(A_n^k\right) = \mu\left(A_n^{(k,n)}\right) = \left[((k,n)\mu_n - 1)/n\right] + 1 \leqslant \mu_k = \mu\left(A_k^k\right)$$

for all $n = 1, 2, \dots,$ and therefore

$$\mu\left(T(M)^k\right) = \mu_k \quad \text{for all } k = 1, 2, \dots.$$

Furthermore, if $r_n \downarrow 0$ fast enough, then $T(M)$ is a nuclear operator. (It suffices to take $r_n = (n + \mu_n)^{-4}$, $n = 1, 2, \dots$.)

This proves Theorem 1 for the case when $\mathscr{X}$ is a Hilbert space. If $\mathscr{X}$ is not a Hilbert space, then it is enough to repeat the above construction with the orthonormal basis replaced by a normalized Markushevich basis (see [**2**]; the details are left to the reader). □

PROOF OF THEOREM 2. We begin by constructing a nuclear normal operator $T(M)$ exactly as in the previous proof.

Let L be a diagonal normal operator defined by $Lf_{ij} = t_i e(v_j) f_{ij}$ with respect to an orthonormal basis $\{f_{ij}\}_{ij=1}^{\infty}$, where $\{t_i\}_{i=1}^{\infty}$ is a denumerable dense subset of (*distinct* points of) $(0,1) \setminus \{r_n\}_{n=1}^{\infty}$ and $\{e(v_j)\}_{j=1}^{\infty}$ is a denumerable dense subset of the unit circle such that v_j and v_j/v_h are irrational for all j and, respectively, for all $h \neq j$. Then L is a normal operator, the set of all eigenvalues of L^k is disjoint from the set of all eigenvalues of $T(M)^k$ for each $k = 1, 2, \dots,$ and it straightforward to check that

$$\begin{aligned} \mu\left(\{T(M) \oplus L\}^k\right) &= \max\left\{\mu\left(T(M)^k\right), \mu\left(L^k\right)\right\}\\ &= \max\{\mu_k, 1\} = \mu_k \quad \text{for all } k = 1, 2, \dots \end{aligned}$$

(see [**1**]), and

$$\sigma(T(M) \oplus L) = \sigma(L) = \{\lambda: |\lambda| \leqslant 1\}.$$

Thus, the normal operator $N(M) = T(M) \oplus L$ satisfies all our requirements.

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