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SUBHOMOGENEOUS AF C^*-ALGEBRAS AND THEIR FUBINI PRODUCTS

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ABSTRACT. We give a characterization of subhomogeneous AF C^*-algebras in terms of their C^*-subalgebras. Also, we show that an AF C^*-algebra is a C^*-algebra with trivial Fubini products if and only if it is subhomogeneous.

1. Introduction. A C^*-algebra A is said to be *subhomogeneous* if all irreducible representations of A are finite dimensional with bounded dimension. Since J. Fell [**5**] initiated the study of such C^*-algebras, subhomogeneous C^*-algebras have been recognized to be an important class of C^*-algebras in the literature [**2**, **8**, **9**, **14–16**]. On the other hand, the structures of approximately finite-dimensional C^*-algebras (AF algebras) have been studied extensively by means of their *associate diagrams* [**3**, **11**].

In this note, we give a characterization of subhomogeneous AF algebras in terms of their C^*-subalgebras. More explicitly, we show that an AF algebra is not subhomogeneous if and only if it contains a special C^*-subalgebra $M \cap K(H)$, which will be described in Example 2.1. Note that every irreducible representation of $M \cap K(H)$ is finite dimensional but can have arbitrarily large dimension.

From T. Huruya's result [**7**] that $M \cap K(H)$ is a C^*-algebra with a nontrivial Fubini product, it is easy to see that an AF algebra is a C^*-algebra with trivial Fubini products only if it is subhomogeneous. This is a partial converse of J. Tomiyama's result [**15**] on trivial Fubini products, for which we also give an alternative short proof.

Throughout this note, we do not assume that AF algebras have identity elements. So all terminologies and notations related to diagrams follow those of A. Lazar and D. Taylor [**11**].

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2. Subhomogeneous AF C^*-algebras.

EXAMPLE 2.1. Let H_n be the n-dimensional Hilbert space for $n = 1, 2, \ldots$ and $H = \oplus_{n=1}^{\infty} H_n$. Also, let $M_n = B(H_n)$ denote the C^*-algebra of all bounded linear

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operators on H_n and $M = \oplus_{n=1}^{\infty} M_n = \{(x_n);\ x_n \in M_n \text{ and } \sup_n\|x_n\| < \infty\}$. If we denote by $K(H)$ the C^*-algebra of all compact operators on H, then the C^*-algebra $M \cap K(H) = \{(x_n);\ x_n \in M_n \text{ and } \lim_n\|x_n\| = 0\}$ is an AF algebra whose associate diagram is as shown in Figure 1. This is T. Huruya's example of the C^*-algebra mentioned in the introduction. Indeed, he showed that $(M \cap K(H)) \otimes_F B(H) \neq (M \cap K(H)) \otimes B(H)$ by using the arguments of S. Wassermann [**17**, **18**] (see §3 for notation).

As a preparation, we give a characterization of subhomogeneous AF algebras in terms of their associate diagrams.

PROPOSITION 2.2. *Let A be an AF algebra with associate diagram $(D, d, \mathscr{U})$. Then the following are equivalent*:

(i) *A is subhomogeneous.*

(ii) *Let $\{X_\lambda: \lambda \in \Lambda\}$ be the family of all connected sequences in D. Then we have*

$$d_\lambda = \sup\{d(x);\ x \in X_\lambda\} < \infty \tag{2.1}$$

for each $\lambda \in \Lambda$, and

$$\sup\{d_\lambda;\ \lambda \in \Lambda\} < \infty. \tag{2.2}$$

PROOF. (i) $\Rightarrow$ (ii). If there exists a connected sequence X_λ in D with $d_\lambda = n$ for some denumerable cardinal number n, then by considering the set of all ancestors of elements of X_λ, we have an n-dimensional irreducible representation of A by [**11**, 2.14].

(ii) $\Rightarrow$ (i). Let $\{\pi, H\}$ be an irreducible representation of A. Then $\pi(A)$ is an AF algebra whose associate diagram $(E, d|_E, \mathscr{U}_E)$ is irreducible [**11**, 2.14] and elementary by (2.1). So, $(E, d|_E, \mathscr{U}_E)$ is equivalent to a subdiagram generated by a complete connected sequence $\{x_k\}$ in E [**11**, 3.6]. Hence, $\pi(A)$ is isomorphic to M_n, where $n = \lim_k d(x_k)$. Therefore, all irreducible representations of A are finite dimensional with bounded dimension by (2.2).

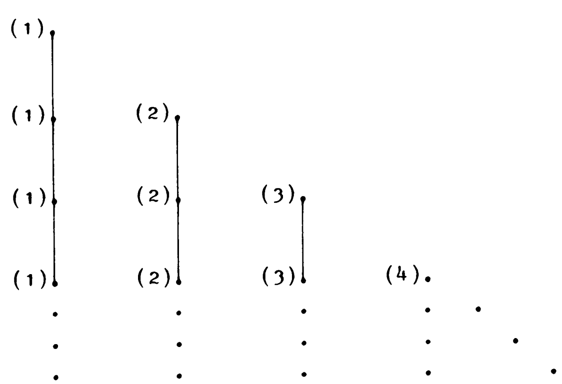

FIGURE 1

Now, we prove our main result. Its proof depends heavily on techniques for dealing with associate diagrams, and the notation introduced in Proposition 2.2 will be used.

THEOREM 2.3. *Let A be an AF algebra with associated diagram $(D, d, \mathcal{U})$. Then the following are equivalent*:

(i) *A is subhomogeneous.*

(ii) *A does not contain a C^*-subalgebra which is isomorphic to the C^*-algebra $M \cap K(H)$.*

PROOF. (i) $\Rightarrow$ (ii). Note that subhomogeneity is inherited by C^*-subalgebras. But, $M \cap K(H)$ is not subhomogeneous.

(ii) $\Rightarrow$ (i). Assume that A is not subhomogeneous. Then, it is sufficient to consider the following three cases:

Case I. A is not postliminal.

By [**10**, 1.1], it is clear that A has an AF subalgebra whose associate diagram is as shown in Figure 1.

Case II. A is postliminal but does not satsify (2.1).

Let $\{x_n\}$ be a connected sequence in D such that $\lim_n d(x_n) = \infty$. Since A is postliminal, it is easy to see that there exists a natural number N such that for every m and n with $m > n \geqslant N$, x_m is a descendant of x_n with multiplicity one [**11**, 3.13]. Hence, by considering a subsequence, we may assume that $d(x_{n+1}) \geqslant d(x_n) + n + 1$ and x_m is a descendant of x_n with multiplicity one for every natural number n and m with $n < m$. Now, we can choose elements $\{u^n_{i,j};\ i, j = 1, 2, \ldots, n\}$ in the factor represented by x_n for $n = 1, 2, \ldots$, with the properties:

$$(2.3)\qquad \begin{cases} \left(u^n_{i,j}\right)^* = u^n_{j,i}, \\ u^n_{i,j}u^m_{k,l} = \delta_{mn}\delta_{jk}u^n_{i,l}, \end{cases} \qquad m, n = 1, 2, \ldots,$$

where δ_{mn} and δ_{jk} are the Kronecker delta functions. Let B_n be the finite-dimensional C^*-subalgebra of A generated by $\{u^m_{i,j};\ i, j = 1, 2, \ldots, m \text{ and } m = 1, 2, \ldots, n\}$ for $n = 1, 2, \ldots$, and B be the norm closure of $\bigcup_{n=1}^{\infty} B_n$. Then, B is an AF subalgebra of A whose associate diagram is as shown in Figure 1.

Case III. A satisfies (2.1) but does not satisfy (2.2).

Choose a sequence $\{\lambda_1, \lambda_2, \ldots\}$ of Λ such that $d_{\lambda_1} < d_{\lambda_2} < \cdots$ and $\lim_n d_{\lambda_n} = \infty$. Put $Y_n = \{x \in X_{\lambda_n};\ d(x) = d_{\lambda_n}\}$. We claim that there exist infinitely many Y_n's with the following property:

$$(2.4)\qquad \begin{array}{l} \text{There exists } x \in Y_n \text{ such that, if } y \in Y_n \text{ is a descendant of } x, \\ \text{then every descendant of } y \text{ in } \bigcup_{i=1}^{\infty} Y_i \text{ lies in } Y_n. \end{array}$$

Indeed, if there exist at most finitely many Y_n's with property (2.4), then there exists a natural number N such that Y_n does not satisfy (2.4) for any $n \geqslant N$. That is, for any $x \in Y_n$ with $n \geqslant N$, there exists a descendant of $y \in Y_n$ of x such that y has a descendant in Y_m for some $m > n$. This implies that there exists a connected sequence $\{x_n\}$ such that $\lim_n d(x_n) = \infty$, contradicting (2.1).

Now, we rename the Y_n's satisfying (2.4) as $\{Z_k;\ k = 1, 2 \ldots\}$, and put $Z'_k = \{y \in Z_k;\ y$ is a descendant of x arising in (2.4)$\}$ for $k = 1, 2, \ldots$. Again, we may choose elements $\{u^k_{i,j};\ i, j = 1, 2, \ldots, k\}$ from a factor in Z'_k satisfying (2.3) for $k = 1, 2, \ldots$. So, A contains a C^*-subalgebra which is isomorphic to $M \cap K(H)$ as in Case II, and this completes the proof.

Let A be a W^*-algebra. Then, A does not contain M as a closed $*$-subalgebra if and only if A is a direct sum of finitely many type I W^*-algebras of the form $Z \otimes M_n$, where $n < \infty$ and Z is an abelian W^*-algebra (see the proof of [**17**, 1.9]). Note that a C^*-algebra A is subhomogeneous if and only if A can be embedded in some C^*-algebra $Z \otimes M_n$, where $n < \infty$ and Z is an abelian C^*-algebra. Hence, Theorem 2.3 can be considered as a C^*-algebraic counterpart of the above fact about W^*-algebras.

Of course, Theorem 2.3 is not valid for arbitrary C^*-algebras because there are several examples of nonsubhomogeneous C^*-algebras which have no nontrivial projections (see [**4**] for example). In fact, our proof depends on the abundance of projections in AF algebras. So, it seems to be an interesting problem to what extent Theorem 2.3 is valid. For example, various conditions on the density of projections by G. Pedersen [**13**] can be considered.

3. AF C^*-algebras with trivial Fubini products. Let C and D be C^*-algebras and $C \otimes D$ the minimal C^*-tensor product of C and D. For $f \in C^*$ (resp. $g \in D^*$), let $R_f\colon C \otimes D \to D$ (resp. $L_g\colon C \otimes D \to C$) be the unique bounded linear map satisfying $R_f(c \otimes d) = f(c)d$ (resp. $L_g(c \otimes d) = g(d)c$). If A and B are C^*-subalgebras of C and D, then the *Fubini product $F(A, B, C \otimes D)$ of A and B with respect to $C \otimes D$* is defined by $F(A, B, C \otimes D) = \{x \in C \otimes D;\ R_f(x) \in B$ and $L_g(x) \in A$ for all $f \in C^*$ and $g \in D^*\}$ [**15**]. Also, T. Huruya [**7**] showed that if C_1 and C_2 (resp. D_1 and D_2) are injective C^*-algebras containing A (resp. B), then $F(A, B, C_1 \otimes D_1)$ and $F(A, B, C_2 \otimes D_2)$ are isomorphic and they are the largest Fubini products, which are denoted by $A \otimes_F B$. Now, J. Tomiyama's result [**15**, 3.1] says that if A is subhomogeneous, then A is a C^*-algebra *with trivial Fubini products*, that is, $A \otimes_F B = A \otimes B$ for every C^*-algebra B. In Theorem 3.2, we show that the converse of J. Tomiyama's result holds for AF algebras by using T. Huruya's example (see Example 2.1) and Theorem 2.3.

If Z is an abelian von Neumann algebra, then $A = Z \otimes M_n = Z \overline{\otimes} M_n$ is an injective von Neumann algebra. Let C be a C^*-algebra containing A and (B, D) a pair of C^*-algebras with $B \subset D$. Then, $F(A, B, C \otimes D) \subset F(A, D, C \otimes D) = A \otimes D$ by [**15**, 3.7], and $F(A, B, C \otimes D) \subset F(A, B, A \otimes D) = A \otimes B$ because A is nuclear. Hence, $A = Z \otimes M_n$ is a C^*-algebra with trivial Fubini products and Lemma 3.1 implies that every C^*-subalgebra of $Z \otimes M_n$ is also a C^*-algebra with trivial Fubini products. This gives an alternative shorter proof for J. Tomiyama's result.

LEMMA 3.1. *Let A be a nuclear C^*-algebra with trivial Fubini products. If C is a nuclear C^*-subalgebra of A, then C is also a C^*-algebra with trivial Fubini products.*

PROOF. Let A_0 be an injective C^*-algebra containing A and (B, D) a pair of C^*-algebras with $B \subset D$. Then, $F(C, B, A_0 \otimes D) \subset F(A, B, A_0 \otimes D) = A \otimes B$. Hence, we have $F(C, B, A_0 \otimes D) \subset F(C, B, A \otimes B) = C \otimes B$ by [**1**, 3.4].

THEOREM 3.2. *Let A be an AF C^*-algebra. Then A is a C^*-algebra with trivial Fubini products if and only if A is subhomogeneous.*

PROOF. It suffices to prove the necessity. If A is not subhomogeneous, then A contains a C^*-subalgebra which is isomorphic to the C^*-algebra $M \cap K(H)$ by Theorem 2.3. But, since $M \cap K(H)$ is a nuclear C^*-algebra which has a nontrivial Fubini product, we have a contradiction by Lemma 3.1.

The author does not know whether the property "with trivial Fubini products" is inherited by C^*-subalgebras and C^*-quotients or not. If these questions can be answered affirmatively, then it can be shown that every irreducible representation of a C^*-algebra with trivial Fubini products is finite dimensional by using J. Glimm's result [**6**] (see also [**12**, 6.7.4]) which says that a C^*-algebra is not postliminal if and only if it contains a C^*-subalgebra which has a UHF quotient.

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