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THE STRONG LIMIT OF VON NEUMANN SUBALGEBRAS WITH CONDITIONAL EXPECTATIONS[1]

MAKOTO TSUKADA

ABSTRACT. The strong lower limit and the weak upper limit of a net of von Neumann subalgebras on which the conditional expectations exist with respect to a fixed faithful normal state are defined. The limits coincide if and only if the corresponding conditional expectations converge strongly.

1. Preliminaries. Let M be a σ-finite von Neumann algebra and φ a faithful normal state on M. By the GNS construction it can be considered that M is acting on a Hilbert space H and there exists a cyclic separating vector $\Phi \in H$ with $\varphi(x) = \langle \Phi | x\Phi \rangle$ for every $x \in M$. Denote by M_* the space of all σ-weakly continuous linear functionals on M. That is, M_* is the predual of M.

For a von Neumann subalgebra N of M, if there exists a projection ε of norm one from M onto N with $\varphi \circ \varepsilon = \varphi$, ε is called the *conditional expectation* onto N [**3**, **6**].

1°. The conditional expectation onto N exists if and only if $\sigma_t(N) = N$ for every $t \in \mathbf{R}$, where $\{\sigma_t\}$ is the modular automorphism group on M with respect to φ.

2°. If the conditional expectation ε onto N exists, then $\varepsilon(x)\Phi = Px\Phi$ for every $x \in M$, where P is the orthogonal projection of H onto $\overline{N\Phi}$.

Throughout this paper we fix a net $\{N_\alpha\}$ of von Neumann subalgebras of M and assume that the conditional expectation ε_α onto N_α exists for each α. The orthogonal projection of H onto $H_\alpha = \overline{N_\alpha \Phi}$ is denoted by P_α. In the recent paper [**5**] we proved that if $\{N_\alpha\}$ is increasing (resp. decreasing), then the conditional expectation ε_∞ onto $\bigvee_\alpha N_\alpha$ (resp. $\bigcap_\alpha N_\alpha$) exists and $\varepsilon_\alpha(x) \to \varepsilon_\infty(x)$ strongly for every $x \in M$ and $f \circ \varepsilon_\alpha \to f \circ \varepsilon_\infty$ in norm for every $f \in M_*$. In this paper we shall introduce the notion of the strong limit of $\{N_\alpha\}$ and show that the limit exists if and only if the corresponding $\{\varepsilon_\alpha\}$ converge strongly. The following are elementary but will be useful below.

3°. For any uniformly bounded net $\{x_\gamma\}$ in M and $x \in M$, $x_\gamma \to x$ strongly (resp. weakly) if and only if $x_\gamma\Phi \to x\Phi$ strongly (resp. weakly) in H.

4°. Let $\{P_\gamma\}$ be a net of orthogonal projections of H, and P an orthogonal projection of H. For any $\xi \in H$, if $P_\gamma\xi \to P\xi$ weakly, then it does strongly.

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2. The strong limit of $\{N_\alpha\}$. We define the *strong lower limit* s-$\liminf N_\alpha$ and the *weak upper limit* w-$\limsup N_\alpha$ of $\{N_\alpha\}$ as follows:

I. $x \in s$-$\liminf N_\alpha$ if and only if there exist $x_\alpha \in N_\alpha$ for each α such that $\sup_\alpha \|x_\alpha\| < \infty$ and $x_\alpha \to x$ strongly;

II. $x \in w$-$\limsup N_\alpha$ if and only if there exist $x_{\alpha'} \in N_{\alpha'}$ for each α', where $\{N_{\alpha'}\}$ is a subnet of $\{N_\alpha\}$, such that $\sup_{\alpha'}\|x_{\alpha'}\| < \infty$ and $x_{\alpha'} \to x$ weakly.

THEOREM 1. (i) s-$\liminf N_\alpha$ *is a von Neumann subalgebra of M and satisfies the following equalities*:

$$\begin{aligned} s\text{-}\liminf N_\alpha &= \{x \in M : \varepsilon_\alpha(x) \to x \textit{ strongly}\} \\ &= \{x \in M : \varepsilon_\alpha(x) \to x \textit{ weakly}\}. \end{aligned}$$

(ii) *Both the conditional expectation onto* s-$\liminf N_\alpha$ *and the one onto* $(w$-$\limsup N_\alpha)''$ *exist.*

(iii)
$$\bigvee_\beta \bigcap_{\alpha \succcurlyeq \beta} N_\alpha \subseteq s\text{-}\liminf N_\alpha \subseteq w\text{-}\limsup N_\alpha \subseteq \bigcap_\beta \bigvee_{\alpha \succcurlyeq \beta} N_\alpha.$$

PROOF. (i) We establish the first equality. "$\supseteq$" is clear. Let x belong to s-$\liminf N_\alpha$. There exists a uniformly bounded net $\{x_\alpha\}$ such that $x_\alpha \in N_\alpha$ for each α and $x_\alpha \to x$ strongly. Then

$$\|\varepsilon_\alpha(x)\Phi - x\Phi\| = \|P_\alpha x\Phi - x\Phi\| \leqslant \|x_\alpha\Phi - x\Phi\| \to 0 \quad (\text{as } \alpha \uparrow).$$

Thus $\varepsilon_\alpha(x) \to x$ strongly by 3° in §1. The second equality follows from $2^\circ, 3^\circ$ and 4° in §1. We next show that s-$\liminf N_\alpha$ is a von Neumann subalgebra of M. It is clearly closed under linear combination. Let x and y belong to s-$\liminf N_\alpha$. Then

$$\begin{aligned} &\|\varepsilon_\alpha(x)\varepsilon_\alpha(y)\Phi - xy\Phi\| \\ &\quad\leqslant \|\varepsilon_\alpha(x)\varepsilon_\alpha(y)\Phi - \varepsilon_\alpha(x)y\Phi\| + \|\varepsilon_\alpha(x)y\Phi - xy\Phi\| \\ &\quad\leqslant \|x\| \cdot \|\varepsilon_\alpha(y)\Phi - y\Phi\| + \|\varepsilon_\alpha(x)y\Phi - xy\Phi\| \to 0 \quad (\text{as } \alpha \uparrow). \end{aligned}$$

Hence $xy \in s$-$\liminf N_\alpha$. On the other hand, since $\varepsilon_\alpha(x) \to x$ weakly, $\varepsilon_\alpha(x^*) \to x^*$ weakly. Therefore $x^* \in s$-$\liminf N_\alpha$. Finally, let x belong to the strong closure of s-$\liminf N_\alpha$. Then for any $\varepsilon > 0$ there exists $y \in s$-$\liminf N_\alpha$ such that $\|x\Phi - y\Phi\| < \varepsilon/3$, and for this y there exists α_0 such that $\|\varepsilon_\alpha(y)\Phi - y\Phi\| < \varepsilon/3$ for any $\alpha \succcurlyeq \alpha_0$. Hence

$$\begin{aligned} \|\varepsilon_\alpha(x)\Phi - x\Phi\| &\leqslant \|P_\alpha x\Phi - P_\alpha y\Phi\| + \|P_\alpha y\Phi - y\Phi\| + \|y\Phi - x\Phi\| \\ &\leqslant 2 \cdot \|x\Phi - y\Phi\| + \|\varepsilon_\alpha(y)\Phi - y\Phi\| < \varepsilon \end{aligned}$$

for any $\alpha \succcurlyeq \alpha_0$. Therefore $\varepsilon_\alpha(x) \to x$ strongly and we have $x \in s$-$\liminf N_\alpha$.

(ii) This follows from 1° in §1. Indeed, let $x \in s$-$\liminf N_\alpha$. Then there exist $x_\alpha \in N_\alpha$ for each α such that $x_\alpha \to x$ strongly. Since $\sigma_t(x_\alpha) \in N_\alpha$ for each α and $\sigma_t(x_\alpha) \to \sigma_t(x)$ strongly, $\sigma_t(x) \in s$-$\liminf N_\alpha$. Thus s-$\liminf N_\alpha$ is globally invariant under the modular automorphism group. Similarly, we have

$$\sigma_t(w\text{-}\limsup N_\alpha) = w\text{-}\limsup N_\alpha.$$

Since σ_t is an automorphism,

$$\sigma_t(w\text{-}\limsup N_\alpha)'' = \sigma_t\big((w\text{-}\limsup N_\alpha)''\big).$$

Thus $(w\text{-}\limsup N_\alpha)''$ is also globally invariant under the automorphism group.

(iii) This is easily verified.

EXAMPLES. (i) Let N_0 and N_1 be von Neumann subalgebras with conditional expectations and $N_{2n} = N_0$ and $N_{2n+1} = N_1$ for $n = 1, 2, \ldots$. Then $s\text{-}\liminf N_n = N_0 \cap N_1$ and $w\text{-}\limsup N_n = N_0 \cup N_1$. Hence, in general, $s\text{-}\liminf N_n \neq w\text{-}\limsup N_n$ and $w\text{-}\limsup N_n$ is not a von Neumann subalgebra.

(ii) If $\{N_\alpha\}$ is increasing (resp. decreasing) and $N_\infty = \bigvee_\alpha N_\alpha$ (resp. $\bigcap_\alpha N_\alpha$), then $\bigvee_\beta \bigcap_{\alpha \succcurlyeq \beta} N_\alpha = \bigcap_\beta \bigvee_{\alpha \succcurlyeq \beta} N_\alpha = N_\infty$. In this case by Theorem 1 (iii),

$$s\text{-}\liminf N_\alpha = w\text{-}\limsup N_\alpha = N_\infty.$$

(iii) Denote by $\mathrm{Aut}_\varphi(M)$ the family of φ-invariant automorphisms on M. Let N be a von Neuman subalgebra with conditional expectation ε. For any $a \in \mathrm{Aut}_\varphi(M)$ the conditional expectation ε_a onto $N_a = a(N)$ exists. Indeed $\varepsilon_a = a \circ \varepsilon \circ a^{-1}$. Let $\{a_\lambda\} \subseteq \mathrm{Aut}_\varphi(M)$ be a net such that $a_\lambda \to a$ strongly as $\lambda \uparrow$ for some $a \in \mathrm{Aut}_\varphi(M)$. Then $\varepsilon_{a_\lambda}(x) \to \varepsilon_a(x)$ strongly as $\lambda \uparrow$ for every $x \in M$. Hence, by Theorem 2

$$s\text{-}\liminf N_{a_\lambda} = w\text{-}\limsup N_{a_\lambda} = N_a.$$

(iv) Let M be the 2×2 matrix algebra, φ the normalized trace on M, and N the set of diagonal matrices in M. Define

$$U_n = \begin{pmatrix} \cos \pi/n & \sin \pi/n \\ -\sin \pi/n & \cos \pi/n \end{pmatrix}$$

and $N_n = U_n^* N U_n$ for $n = 1, 2, \ldots$. Then

$$s\text{-}\liminf N_n = w\text{-}\limsup N_n = N,$$

but $\bigvee_m \bigcap_{n \geqslant m} N_n = \mathbf{C} \cdot 1$ and $\bigcap_m \bigvee_{n \geqslant m} N_n = M$.

(v) Let N_1 and N_2 be von Neumann subalgebras with conditional expectations, S_1 (resp. S_2) the unit balls of N_1 (resp. N_2), and P_1 (resp. P_2) the orthogonal projections onto $\overline{N_1\Phi}$ (resp. $\overline{N_2\Phi}$). Define

$$d(N_1, N_2) = \max\left\{ \sup_{x \in S_1} \|x\Phi - P_2 x\Phi\|, \sup_{x \in S_2} \|x\Phi - P_1 x\Phi\| \right\}.$$

This is the so-called Hausdorff distance between $S_1\Phi$ and $S_2\Phi$. Suppose that there exists a von Neumann subalgebra N with conditional expectation ε such that $d(N_\alpha, N) \to 0$ as $\alpha \uparrow$. Let P be the orthogonal projection onto $\overline{N\Phi}$. Then for any $\xi \in H$ with $\|\xi\| \leqslant 1$

$$\begin{aligned} \|P\xi - P_\alpha\xi\|^2 &= \langle P\xi|\xi\rangle - \langle P_\alpha P\xi|\xi\rangle - \langle PP_\alpha\xi|\xi\rangle + \langle P_\alpha\xi|\xi\rangle \\ &\leqslant \|P\xi - P_\alpha P\xi\| + \|PP_\alpha\xi - P_\alpha\xi\| \\ &\leqslant 2 \cdot d(N_\alpha, N) \to 0 \quad (\text{as } \alpha \uparrow). \end{aligned}$$

Therefore $P_\alpha \to P$ strongly, and by 2° and 3° in §1 we have $\varepsilon_\alpha(x) \to \varepsilon(x)$ strongly for every $x \in H$. Thus, by Theorem 2

$$s\text{-}\liminf N_\alpha = w\text{-}\limsup N_\alpha = N.$$

The converse is not always true. Let $\{N_\alpha\}$ be strictly increasing and $N = \bigvee_\alpha N_\alpha$. Then for any α there exists $\xi \in \overline{N\Phi}$ with $\|\xi\| = 1$ and $\xi \perp \overline{N_\alpha \Phi}$. Hence $d(N_\alpha, N) = 1$ for every α, and N_α does not converge to N in the Hausdorff metric topology, but

$$s\text{-}\liminf N_\alpha = w\text{-}\limsup N_\alpha = N.$$

3. Strong convergence of $\{\varepsilon_\alpha\}$. We now state our main theorem.

THEOREM 2. *The following assertions are equivalent*:

(i) s-$\liminf N_\alpha = w$-$\limsup N_\alpha$;

(ii) *there exists a conditional expectation* ε_∞ *such that* $\varepsilon_\alpha(x) \to \varepsilon_\infty(x)$ *weakly for every* $x \in M$;

(iii) $\{\varepsilon_\alpha(x)\}$ *is a strongly convergent net for every* $x \in M$;

(iv) $\{f \circ \varepsilon_\alpha\}$ *is a convergent net in norm for every* $f \in M_*$.

Moreover, if the above assertions are satisfied, then ε_∞ *in* (ii) *is the conditional expectation onto* s-$\liminf N_\alpha$, $\varepsilon_\alpha(x) \to \varepsilon_\infty(x)$ *strongly, and* $f \circ \varepsilon_\alpha \to f \circ \varepsilon_\infty$ *in norm.*

PROOF. (i) $\Rightarrow$ (ii). Let $N_\infty = s$-$\liminf N_\alpha$. The conditional expectation onto N_∞ and the orthogonal projection onto $\overline{N_\infty \Phi}$ are denoted by ε_∞ and P_∞, respectively. Fix $x \in M$. Since $\{\varepsilon_\alpha(x)\}$ is uniformly bounded, for any subnet $\{\varepsilon_{\alpha'}(x)\}$ of $\{\varepsilon_\alpha(x)\}$ there exists its subnet $\{\varepsilon_{\alpha''}(x)\}$ which converges to some $y \in M$ weakly. By the assumption $y \in N_\infty$. For any $z \in N_\infty$

$$\begin{aligned}\|x\Phi - y\Phi\| &\leqslant \liminf \|x\Phi - P_{\alpha''} x\Phi\| \\ &\leqslant \lim \|x\Phi - P_{\alpha''} z\Phi\| = \|x\Phi - z\Phi\|,\end{aligned}$$

because $P_{\alpha''} x\Phi \to y\Phi$ weakly and the norm of H is weakly lower semicontinuous. Since $z \in N_\infty$ is arbitrary, we have $y\Phi = P_\infty x\Phi = \varepsilon_\infty(x)\Phi$. Thus $\varepsilon_\alpha(x) \to \varepsilon_\infty(x)$ weakly.

(ii) $\Rightarrow$ (iii). It follows from 2°, 3° and 4° in §1.

(iii) $\Rightarrow$ (iv). Let $f \in M_*$ be fixed. It can be assumed without loss of generality that f is positive. Then there exists $\xi \in H$ such that $f(x) = \langle \xi | x\xi \rangle$ for every $x \in M$ (see [1, Theorem 6]). For any α and β

$$\begin{aligned}\|f \circ \varepsilon_\alpha - f \circ \varepsilon_\beta\| &\leqslant \sup_{\|x\| \leqslant 1} \left|\langle \xi | \varepsilon_\alpha(x)\xi \rangle - \langle \xi | \varepsilon_\beta(x)\xi \rangle\right| \\ &\leqslant \sup_{\|x\| \leqslant 1} \left|\langle P_\alpha \xi | x\xi \rangle - \langle P_\beta \xi | x\xi \rangle\right| \\ &\leqslant \|P_\alpha \xi - P_\beta \xi\| \cdot \|\xi\|.\end{aligned}$$

Since $\{P_\alpha \xi\}$ is a Cauchy net, so is $\{f \circ \varepsilon_\alpha\}$ and we have (iv).

(iv) $\Rightarrow$ (i). Now we denote by ε_α^* the operator $f \mapsto f \circ \varepsilon_\alpha$ on M_*. Then ε_α^* is a norm-one projection (see [5]). Putting $\varepsilon_\infty^*(f) = s$-$\lim \varepsilon_\alpha^*(f)$ $(f \in M_*)$, ε_∞^* is a bounded linear operator on M_*. Furthermore, for any α

$$\begin{aligned}\|\varepsilon_\infty^* \circ \varepsilon_\infty^*(f) - \varepsilon_\infty^*(f)\| &\leqslant \|\varepsilon_\infty^* \circ \varepsilon_\infty^*(f) - \varepsilon_\alpha^* \circ \varepsilon_\infty^*(f)\| \\ &\quad + \|\varepsilon_\alpha^* \circ \varepsilon_\infty^*(f) - \varepsilon_\alpha^* \circ \varepsilon_\alpha^*(f)\| + \|\varepsilon_\alpha^*(f) - \varepsilon_\infty^*(f)\| \\ &\leqslant \|\varepsilon_\infty^* \circ \varepsilon_\infty^*(f) - \varepsilon_\alpha^* \circ \varepsilon_\infty^*(f)\| + 2 \cdot \|\varepsilon_\infty^*(f) - \varepsilon_\alpha^*(f)\|,\end{aligned}$$

so that ε_∞^* is idempotent. We denote by ε_∞ the conjugate operator of ε_∞^* on M. Then $\varepsilon_\infty(x) = w^*\text{-lim}\,\varepsilon_\alpha(x)$ for every $x \in M$ and ε_∞ is also idempotent. We put $P_\infty x\Phi = \varepsilon_\infty(x)\Phi$ $(x \in M)$. Then $P_\infty x\Phi = w\text{-lim}\, P_\alpha x\Phi$ and

$$\|P_\infty x\Phi\| \leqslant \liminf \|P_\alpha x\Phi\| \leqslant \|x\Phi\|$$

for every $x \in M$. Hence P_∞ is extended to a bounded linear operator on H. Since ε_∞ is idempotent, so is P_∞. Since

$$\langle P_\infty x\Phi | y\Phi\rangle = \lim \langle P_\alpha x\Phi | y\Phi\rangle = \lim \langle x\Phi | P_\alpha y\Phi\rangle = \langle x\Phi | P_\infty y\Phi\rangle$$

for every x, $y \in M$, P_∞ is Hermitian. Therefore P_∞ is an orthogonal projection on H. By 4° in §1 we have $P_\alpha \to P_\infty$ strongly

Now fix $x \in w\text{-lim sup}\, N_\alpha$. Then there exists a uniformly bounded net $\{x_{\alpha'}\}$ such that $x_{\alpha'} \in N_{\alpha'}$ for each α', where $\{N_{\alpha'}\}$ is a subnet of $\{N_\alpha\}$, and $x_{\alpha'} \to x$ weakly. For any $\xi \in H$

$$\langle P_{\alpha'} x\Phi - x_{\alpha'}\Phi | \xi\rangle = \langle x\Phi - x_{\alpha'}\Phi | P_{\alpha'}\xi\rangle \to 0 \quad (\text{as } \alpha' \uparrow),$$

because $\{x\Phi - x_{\alpha'}\Phi\}$ is uniformly bounded and $P_{\alpha'} \to P_\infty$ strongly. Therefore, we have $P_\infty x\Phi = x\Phi$ and $\varepsilon_\alpha(x) \to x$ strongly. Thus $x \in s\text{-lim inf}\, N_\alpha$ and (i) is proved.

If a net $\{x_\alpha\}$ in M satisfies that for any α there exists $y_\alpha \in M$ such that $x_\beta = \varepsilon_\beta(y_\alpha)$ for any $\beta \preccurlyeq \alpha$, then $\{x_\alpha\}$ is called a *martingale* dominated by $\{y_\alpha\}$. A net $\{f_\alpha\}$ in M_* is also called a martingale dominated by $\{g_\alpha\}$, if $f_\beta = g_\alpha \circ \varepsilon_\beta$ for any $\beta \preccurlyeq \alpha$.

THEOREM 3. (i) *Let $\{x_\alpha\} \subseteq M$ be a martingale dominated by $\{y_\alpha\}$. If $\{y_\alpha\}$ is uniformly bounded, then there exists $x \in M$ such that $x_\alpha = \varepsilon_\alpha(x)$ for every α.*

(ii) *Let $\{f_\alpha\} \subseteq M_*$ be a martingale dominated by $\{g_\alpha\}$. If $\{g_\alpha\}$ is weakly relatively compact, then there exists $f \in M_*$ such that $f_\alpha = f \circ \varepsilon_\alpha$ for every α.*

PROOF. Since $\{y_\alpha\}$ is uniformly bounded, there exists a subnet $\{y_{\alpha'}\}$ of $\{y_\alpha\}$ such that $y_{\alpha'} \to x$ σ-weakly for some $x \in M$. Because any conditional expectation is σ-weakly continuous, for any fixed α, $\varepsilon_\alpha(y_{\alpha'}) \to \varepsilon_\alpha(x)$ σ-weakly as $\alpha' \uparrow$. On the other hand, for sufficiently large α' we have $\varepsilon_\alpha(y_{\alpha'}) = x_\alpha$, so that $\varepsilon_\alpha(x) = x_\alpha$. Thus (i) is proved.

(ii) is also proved similarly, because $\psi \mapsto \psi \circ \varepsilon_\alpha$ is weakly continuous on M_* for every α.

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Department of Information Sciences, Science University of Tokyo, Noda City, Chiba 278, Japan