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QUASISIMILAR OPERATORS IN THE COMMUTANT OF A CYCLIC SUBNORMAL OPERATOR

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ABSTRACT. Compactly supported positive regular Borel measures on the complex plane that share "certain" properties with normalized arclength measure on the boundary of the unit disk are called m-measures. Let μ be an m-measure and let S_μ be the cyclic subnormal operator of multiplication by z on the closure of the polynomials in $L^2(\mu)$. Necessary and sufficient conditions for an operator in the commutant of S_μ to be quasisimilar to S_μ are investigated. In particular it is shown that if the Bergman shift and an operator in its commutant are quasisimilar, then they are unitarily equivalent.

1. Introduction. The canonical model for a cyclic subnormal operator is the operator S_μ of multiplication by z on $P^2(\mu)$, the closure of the analytic polynomials in $L^2(\mu)$. Here μ is any compactly supported positive regular Borel measure on the complex plane $\mathbf{C}$. From now on such a μ will be referred to as a *measure*. The problem of determining quasisimilarity between cyclic subnormal operators is, in general, difficult. However, W. S. Clary [**1**] has given necessary and sufficient conditions for a subnormal operator to be quasisimilar to the unilateral shift, and J. B. Conway [**4**] has shown that if the unilateral shift and an operator in its commutant are quasisimilar, then they are unitarily equivalent.

This paper investigates the equivalence relation of quasisimilarity between S_μ and $\phi(S_\mu)$ when μ is a measure with many of the same properties as *normalized arclength measure* m on the *boundary of the open unit disk* $\mathbf{D}$, $\phi \in P^2(\mu) \cap L^\infty(\mu)$, and S_μ is the operator of multiplication by z on $P^2(\mu)$. The reader should bear in mind Yoshino's Theorem [**9**] that states if μ is a measure, then the commutant of S_μ is $\{\phi(S_\mu): \phi \in P^2(\mu) \cap L^\infty(\mu)\}$.

The formal definition of an m-measure is now stated.

1.1 DEFINITION. An *m-measure* is a measure μ on $\mathbf{C}$ such that

(a) the identity mapping on the polynomials extends to an isometric isomorphism that is a weak*-homeomorphism from $P^\infty(\mu)$, the weak*-closure of the polynomials in $L^\infty(\mu)$, onto $P^\infty(m)$;

(b) for every λ in the open disk $\mathbf{D}$, there exist $r > 0$ and $C > 0$ such that

$$|p(z)| \leqslant C\left(\int |p|^2\, d\mu\right)^{1/2}$$

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for all polynomials p and for every $z \in B(\lambda; r)$, the open ball of radius r centered at λ.

If ν is a probability measure on $[0,1]$ with 1 in the support of ν and $d\mu(re^{i\theta}) = dm(e^{i\theta})d\nu(r)$, then μ is an m-measure and S_μ is a subnormal weighted shift of norm 1 [**3**, Theorem III.8.16, Proposition III.8.19]. Other examples of m-measures may be found in [**7**]. We remark that condition (a) in Definition 1.1 is frequently stated as "the Sarason hull of μ is **D**, and $P^\infty(\mu)$ has no L^∞-summand". A detailed discussion of $P^\infty(\mu)$ may be found in [**3**]. Condition (b) is more succinctly phrased as "the analytic bounded point evaluations of $P^2(\mu)$ is **D**", ($B_a(\mu) = \mathbf{D}$). Condition (b) guarantees, for every $\lambda \in \mathbf{D}$, the existence of $k_\lambda \in P^2(\mu)$ such that the function f: $\mathbf{D} \to \mathbf{C}$, given by $\tilde{f}(\lambda) = \int f\bar{k}_\lambda \, d\mu$, is analytic on **D** for every $f \in P^2(\mu)$. Since $\tilde{f} = f$ μ-a.e. on **D**, *we will always assume that f is defined pointwise on* **D**.

The question that this paper will be concerned with is the following.

1.2 QUESTION. *Given an m-measure μ, what are necessary and sufficient conditions on $\phi \in P^2(\mu) \cap L^\infty(\mu)$ for $\phi(S_\mu)$ to be quasisimilar to S_μ?*

In view of the subsequent theorem and Proposition 2.5 of [**2**], it is enough to consider the case of an m-measure μ such that $P^2(\mu) \cap L^\infty(\mu) = P^\infty(\mu)$. In such a case we will say that μ is a *reduced m-measure*. Note that if μ is an m-measure, then $P^\infty(\mu) = P^\infty(m) = H^\infty$ (the set of bounded analytic functions on **D**).

1.3 THEOREM (RAPHAEL [**7**]). *If μ is an m-measure, then there exists a unique m-measurable subset $\Delta \subseteq \partial\mathbf{D}$ such that*

(i) $P^2(\mu) \cap L^\infty(\mu) = P^\infty(\mu|\mathbf{C} \setminus \Delta) \oplus L^\infty(\mu|\Delta)$;

(ii) $\mu|(\mathbf{C} \setminus \Delta)$ *is a reduced m-measure.*

2. Results. We begin with a proposition that characterizes quasiaffinities on $P^2(\mu)$ that intertwine S_μ and $\phi(S_\mu)$.

2.1 PROPOSITION. *Let μ be a reduced m-measure. If $\phi \in H^\infty$ and X: $P^2(\mu) \to P^2(\mu)$ is a one-to-one bounded linear operator with dense range such that $XS_\mu = \phi(S_\mu)X$, then*

(a) $\phi(\mathbf{D}) \subseteq \mathbf{D}$ *and ϕ is one-to-one;*

(b) *there exists a cyclic vector $\tau \in P^2(\mu)$ for $\phi(S_\mu)$ such that*

$$(Xf)(z) = \begin{cases} (f \circ \phi)(z) \cdot \tau(z) & \text{if } z \in \mathbf{D}, \\ (f \circ \phi)(z) \cdot \tau(z) \ \mu\text{-a.e.} & \text{on } \partial\mathbf{D} \end{cases}$$

for every $f \in P^2(\mu)$.

PROOF. Using Corollary 2.6 of [**7**] and Theorem 3.1 of [**1**] it follows that $\phi(\mathbf{D})^- = \sigma(\phi(S_\mu)) \subseteq \sigma(S_\mu) = \mathbf{D}^-$. Since ϕ cannot be a constant function, $\phi(\mathbf{D}) \subseteq \mathbf{D}$. Let $\tau = X1$. Then $\tau \in P^2(\mu)$ and τ is a cyclic vector for $\phi(S_\mu)$. Also

$$Xp = Xp\big(\phi(S_\mu)\big)1 = p\big(\phi(S_\mu)\big)X1 = p(\phi) \cdot \tau \tag{2.2}$$

for all polynomials p. Given $f \in P^2(\mu)$ let $\{p_n\}$ be a sequence of polynomials such that $\int |p_n - f|^2 \, d\mu \to 0$, $p_n \to f$ μ-a.e., and $Xp_n \to Xf$ μ-a.e. Conclusion (b) now follows by taking limits in (2.2).

To show that ϕ is one-to-one on $\mathbf{D}$ let $\alpha, \beta \in \mathbf{D}$ such that $\phi(\alpha) = \phi(\beta)$. For any polynomial p,

$$(Xp)(\alpha) = p(\phi(\alpha)) \cdot \tau(\alpha)$$

and

$$(Xp)(\beta) = p(\phi(\alpha)) \cdot \tau(\beta).$$

Choosing a sequence of polynomials $\{p_n\}$ such that $Xp_n \to 1$ in $P^2(\mu)$, we see that $\tau(\alpha) = \tau(\beta)$. Choosing a sequence of polynomials $\{q_n\}$ such that $Xq_n \to z + 2$ in $P^2(\mu)$, we see that $\tau(\alpha)/\tau(\beta) = (\alpha + 2)/(\beta + 2)$; so $\alpha = \beta$. ∎

2.3 COROLLARY. *Let μ be a reduced m-measure. If $\phi \in H^\infty$ and S_μ is quasisimilar to $\phi(S_\mu)$, then there exist $a \in \mathbf{D}$ and $\alpha \in \mathbf{R}$ such that $\phi(z) = e^{i\alpha}(z - a)/(1 - \bar{a}z)$.*

PROOF. It suffices to show that $\phi(\mathbf{D}) = \mathbf{D}$. Suppose for contradiction that there exists $w \in \mathbf{D} \setminus \phi(\mathbf{D})$. Since $\sigma(\phi(S_\mu)) = \sigma(S_\mu) = \mathbf{D}^-$ [**1**, Corollary 3.1A], there exists a sequence $\{b_n\} \subseteq \mathbf{D}$ such that $\phi(b_n) \to w$. Without loss of generality it may be assumed that $b_n \to b \in \partial\mathbf{D}$. By Theorem III.12 of [**5**], w belongs to the essential spectrum of $\phi(S_\mu)$. This is a contradiction, since by Corollary 1.2 of [**8**] the essential spectrum of S_μ is $\partial\mathbf{D}$; but S_μ and $\phi(S_\mu)$ have equal essential spectra [**6**, Corollary 6]. ∎

The next theorem implies that operators obtained by applying conformal maps from $\mathbf{D}$ onto $\mathbf{D}$ to S_μ, when μ is a reduced m-measure, are not always quasisimilar. To see this, take $\mu_k = \lambda + \chi_{J_k} m$ and $\phi(z) = -z$, where λ = area measure on $\mathbf{D}$ and $J_k = (-1)^k\{e^{i\theta}: 0 < \theta < \pi\}$. Then observe that $\phi(S_{\mu_1})$ is unitarily equivalent to S_{μ_2}. (That μ_k is a reduced m-measure is a consequence of Theorem 3.2 of [**8**].)

2.4 THEOREM. *If μ_1 and μ_2 are (arbitrary) measures such that S_{μ_1} is quasisimilar to S_{μ_2} and $\sigma(S_{\mu_1}) \subseteq \mathbf{D}^-$, then $\mu_1|\partial\mathbf{D}$ and $\mu_2|\partial\mathbf{D}$ are mutually absolutely continuous.*

PROOF. Without loss of generality, we may suppose that there exist bounded, linear, one-to-one mappings $X^1_{2,1}$: $P^2(\mu_1) \to P^2(\mu_2)$ and $X^\phi_{1,2}$: $P^2(\mu_2) \to P^2(\mu_1)$ given by $X^1_{2,1}p = p$ and $X^\phi_{1,2}p = p \cdot \phi$ for all polynomials p. Here $\phi \in P^2(\mu_2) \cap L^\infty(\mu_2)$ is a cyclic vector for S_{μ_2} (cf. [**2**, Proposition 4.1]). Then

$$\int |z^n p\phi|^2 \, d(\mu_2|\partial\mathbf{D}) \leqslant \int |z^n p\phi|^2 \, d\mu_2 \leqslant \|X^\phi_{1,2}\|^2 \int |z^n p|^2 \, d\mu_1.$$

Letting $n \to \infty$ we see that $\int |p\phi|^2 \, d(\mu_2|\partial\mathbf{D}) \leqslant \|X^\phi_{1,2}\|^2 \int |p|^2 \, d(\mu_2|\partial\mathbf{D})$. Similarly,

$$\int |p|^2 \, d(\mu_1|\partial\mathbf{D}) \leqslant \|X^1_{2,1}\|^2 \int |p|^2 \, d(\mu_2|\partial\mathbf{D}).$$

By Lemma 2.4 of [**2**], $S_{\mu_1|\partial\mathbf{D}}$ and $S_{\mu_2|\partial\mathbf{D}}$ are quasisimilar.

If $S_{\mu_1|\partial\mathbf{D}}$ is normal, then so is $S_{\mu_2|\partial\mathbf{D}}$, and the result follows from Theorem II.4.6 of [**3**]. If $S_{\mu_1|\partial\mathbf{D}}$ is not normal, then the result follows from [**1**, Lemma 4.5β, Theorem 4.5; and **3**, Proposition III, 14.13]. ∎

The author would like to point out that Theorem 2.4 adds positive evidence to a question raised in [**7**]. Namely, does S_{μ_1} quasisimilar to S_{μ_2} imply that $\mu_1|\sigma_{ap}(S_{\mu_1})$ and $\mu_2|\sigma_{ap}(S_{\mu_2})$ are mutually absolutely continuous?

We conclude with a result concerning the Bergman shift.

2.5 THEOREM. *Let λ be area measure on $\mathbf{D}$. If $\phi \in H^\infty$, then the following are equivalent*:

(i) S_λ *and* $\phi(S_\lambda)$ *are unitarily equivalent.*

(ii) S_λ *and* $\phi(S_\lambda)$ *are quasisimilar.*

(iii) $\phi(z) = e^{i\alpha}(z - a)/(1 - \bar{a}z)$ *for some* $a \in \mathbf{D}$ *and some* $\alpha \in \mathbf{R}$.

PROOF. Since (i) obviously implies (ii), and (ii) implies (iii) by Corollary 2.3, it must be shown that (iii) implies (i).

If $\phi(z) = e^{i\alpha}(z - a)/(1 - \bar{a}z)$ then $\phi'(z) = e^{i\alpha}(1 - |a|^2)/(1 - \bar{a}z)^2$. It follows by the change of variables formula that

$$\int |p(\phi)|^2 |\phi'|^2 \, d\lambda = \int |p|^2 \, d\lambda$$

for all polynomials p. Hence there exists an isomorphism $X: P^2(\lambda) \to P^2(\lambda)$ such that $Xp = p(\phi) \cdot \phi'$. (X is onto since ϕ is a weak*-generator for H^∞ and ϕ' is bounded above and below on $\mathbf{D}^-$.) Since it is easy to see that $XS_\lambda = \phi(S_\lambda)X$, the proof is complete. ■

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