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THE FACTORIZATION OF A LINEAR CONJUGATE SYMMETRIC INVOLUTION IN HILBERT SPACE

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ABSTRACT. Let X be a closed linear transformation whose domain is dense in the complex separable Hilbert space H and whose adjoint is denoted by X^*. The operator X is said to be conjugate symmetric if $\Gamma(X) \subset \Gamma(QX^*Q)$, where $\Gamma(X)$ represents the graph of X in $H \oplus H$ and Q is a conjugation on H. The main theorem in this note states that a conjugate symmetric linear involution X satisfies the operator equation $X = QX^*Q$.

1. Introduction. An *involution* is a transformation of a set into itself whose square is the identity. In this note we shall study linear involutions defined on a dense subset of a complex, infinite dimensional Hilbert space. If the involution is closed and has a polar decomposition of the form $X = U|X|$, then U itself is an involution. This result is exploited to show that any conjugate symmetric involution must be conjugate selfadjoint.

In what follows H will represent a complex Hilbert space having a countably infinite basis with (f, g) being the inner product of two vectors in H. A *manifold* is a subset which is closed under vector addition and under multiplication by complex numbers. A *subspace* is a manifold which is closed in the norm topology induced by the inner product. The closure of a set F in the norm topology will be denoted by clos F, and $F^\perp = \{g|(g, f) = 0, f \in F\}$. The Hilbert space $H \oplus H$ consists of all vectors $f \oplus g$ having inner product

$$(f_1 \oplus g_1, f_2 \oplus g_2) = (f_1, f_2) + (g_1, g_2).$$

If X is a linear transformation defined on a manifold $D \subset H$, its graph

$$\Gamma(X) = \{f \oplus Xf | f \in D\}$$

is another manifold in $H \oplus H$. If this manifold is a *subspace*, then X is called a *closed operator*. If clos $D = H$, then X is said to be *densely defined*. When X is densely defined, it has a unique adjoint X^* determined by the condition $(Xf, g) = (f, X^*g)$ which subsists for all f in Dom X. The domain of X^* is dense in H if and only if X has a closed linear extension. If $\Gamma(X) \subset \Gamma(Y)$, then $\Gamma(Y)^\perp \subset \Gamma(X)^\perp$, and this implies $\Gamma(Y^*) \subset \Gamma(X^*)$. It is not difficult to establish the inclusion $\Gamma(Y^*X^*) \subset \Gamma((XY)^*)$ whenever X, Y, and XY are all densely defined. Moreover, $(XY)^* = Y^*X^*$ when X is a bounded linear transformation [7, p. 301]. A densely defined

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transformation X is said to be *symmetric* if $\Gamma(X) \subset \Gamma(X^*)$. It is called *selfadjoint* if $X = X^*$. The crucial idea of representing a linear transformation through its associated graph subspace first originated in the work of J. von Neumann [**5**].

The next theorem plays a fundamental role in our analysis of closed involutions. Different proofs are presented in [**1** and **3**].

POLAR DECOMPOSITION THEOREM. *Let X be a closed linear operator whose domain is dense in H. Then there is a positive selfadjoint operator $|X|$ whose domain coincides with that of X, and a partial isometry U with initial space* $(\operatorname{Ker} X)^{\perp}$ *and final space* $\operatorname{Clos}(\operatorname{Ran} X)$ *such that $X = U|X|$. This factorization is unique under the additional assumption that* $\operatorname{Ker}|X| = \operatorname{Ker} X$.

2. Involutions. It is self-evident from the definition that an involution on a set S is a one-to-one transformation that maps S onto itself and satisfies the relation $X = X^{-1}$. If we define $V(f \oplus g) = g \oplus f$, then X is an involution if and only if

$$V(\Gamma(X)) \subset \Gamma(X).$$

This invariance principle leads to a simple characterization of closed, densely defined linear involutions.

LEMMA 1. *If X is a closed linear involution whose domain D is dense in H, there are subspaces H^+ and H^- such that $H^+ \cap H^- = \{0\}$ and $D = H^+ + H^-$. The involution X is defined by $X(h^+ + h^-) = h^+ - h^-$ for all h^+ in H^+ and h^- in H^-. Conversely, with every pair of subspaces H^+ and H^- such that $H^+ \cap H^- = \{0\}$ and* $\operatorname{Clos}(H^+ + H^-) = H$, *there is associated a unique densely defined closed linear involution satisfying $X(h^+ + h^-) = h^+ - h^-$ on $D = H^+ + H^-$.*

PROOF. Let X be a closed, densely defined linear involution whose domain is D. Since V is a unitary involution, the restriction of V to the subspace $\Gamma(X)$ is a bounded selfadjoint operator whose spectrum is contained in the two point set $\{-1, 1\}$. If J denotes the restriction of V to the subspace $\Gamma(X)$, then the subspaces $H_0^+ = \{f \oplus g | J(f \oplus g) = f \oplus g\}$ and $H_0^- = \{f \oplus g | J(f \oplus g) = -(f \oplus g)\}$ are mutually orthogonal. The spectral theorem for selfadjoint operators further asserts that $\Gamma(X) = H_0^+ \oplus H_0^-$. If we define $H^+ = P(H_0^+)$ and $H^- = P(H_0^-)$, where $P(f \oplus g) = f$, it is easy to see that $X(h^+ + h^-) = h^+ - h^-$ on $D = H^+ + H^-$. Clearly, both H^+ and H^- are subspaces.

Now suppose that H^+ and H^- are two subspaces such that $H^+ \cap H^- = \{0\}$ and $\operatorname{Clos}(H^+ + H^-) = H$. Define the transformation X by the equation

$$X(h^+ + h^-) = h^+ - h^-.$$

If $h_n^+ + h_n^- \to f$ and $h_n^+ - h_n^- \to g$, then the vector $(f + g)/2$ belongs to H^+ and the vector $(f - g)/2$ belongs to H^-. Hence $X(f) = g$, and we conclude that X is a closed, densely defined involution uniquely associated with the subspaces H^+ and H^-.

THEOREM 1. *Let X be a closed, densely defined linear involution whose polar representation has the form $X = U|X|$. Then U is a unitary involution.*

PROOF. Since X is an involution, $\operatorname{Ker}|X| = \{0\}$ and U is a unitary operator on H. From the identity $X = X^{-1}$ we immediately obtain $U|X| = |X|^{-1}U^*$. Therefore $U|X|U = |X|^{-1}$ and, after taking adjoints, we get $U^*|X|U^* = |X|^{-1}$. Hence $U^2|X|U^2 = |X|$, from which it follows that $U^2|X| = |X|U^{*2}$ because U is a unitary operator. This last equation immediately implies $U^2|X|$ is selfadjoint. Now let $H^+ = \{h^+ | X(h^+) = h^+\}$ and $H^- = \{h^- | X(h^-) = -h^-\}$. Then $|X|h^+ = U^*h^+$, $|X|h^- = -U^*h^-$, and

$$\begin{aligned}\left(U^2|X|(h^+ + h^-), h^+ + h^-\right) &= \left(|X|(h^+ + h^-), U^{*2}(h^+ + h^-)\right)\\ &= \left(U^*(h^+ - h^-), U^{*2}(h^+ + h^-)\right)\\ &= \left(h^+ - h^-, |X|(h^+ - h^-)\right).\end{aligned}$$

Thus $(U^2|X|(h^+ + h^-), h^+ + h^-) \geqslant 0$ for all h^+ in H^+ and h^- in H^-. Since $\operatorname{Dom} X = H^+ + H^-$ by virtue of Lemma 1, $U^2|X|$ is a positive selfadjoint operator on Dom X. The uniqueness of the polar decomposition implies $U^2 = I$.

3. Conjugate symmetric involutions. A transformation Q on H that satisfies $Q(\alpha f + \beta g) = \alpha^* Q(f) + \beta^* Q(g)$ is called a *conjugation* if $Q^2 = I$ and $(Qf, Qg) = (g, f)$ for every f and g in H. We remark in passing that theoretical physicists have found conjugation operators to be useful tools for studying time reversal symmetry in quantum mechanics [**2**, p. 187]. If there is a conjugation Q such that $X = QX^*Q$, we shall call X *conjugate selfadjoint*. If $\Gamma(X) \subset \Gamma(QX^*Q)$, then X is said to be *conjugate symmetric*. Commutativity properties of bounded conjugate selfadjoint operators are treated in [**4**], but little appears to be known about their general structure.

LEMMA 2. *Let X be a closed, densely defined linear involution having the property that $\Gamma(X) \subset \Gamma(QX^*Q)$. If $X = U|X|$ is the polar representation for X, then* $UQ(\operatorname{Dom} X) \subset \operatorname{Dom} X$ *and* $\operatorname{Dom}|X| \subset \operatorname{Dom} Q|X|^{-1}Q$.

PROOF. An application of Theorem 1 immediately yields the identity $UXU = X^*$. Therefore $\Gamma(X) \subset \Gamma(QUXUQ)$, and it follows that $UQ(\operatorname{Dom} X) \subset \operatorname{Dom} X$. Since $\operatorname{Dom} X = \operatorname{Dom}|X|$ and $Q|X|^{-1}Q = QU|X|UQ$, we further have $\operatorname{Dom}|X| \subset \operatorname{Dom}(Q|X|^{-1}Q)$.

LEMMA 3. *Let X denote the conjugate symmetric involution introduced in the previous lemma, and let $V = Q|X|^{-1}Q|X|^{-1}$. Then V is a densely defined linear transformation having the property that* $\operatorname{Dom} V = \operatorname{Dom} X^*$ *and* $\Gamma(V) \subset \Gamma(QUQU)$.

PROOF. Substituting $X^* = U|X|^{-1}$ in the defining equation for V, we get $V = QUX^*QUX^*$. This immediately implies that $\operatorname{Dom} V \subset \operatorname{Dom} X^*$. But $\Gamma(X) \subset \Gamma(QX^*Q)$ by hypothesis, so we have $\Gamma(QUQUX^{*2}) = \Gamma(QUQXUX^*) \subset \Gamma(V)$. Since X^{*2} is the identity on $\operatorname{Dom} X^*$, it follows that $\operatorname{Dom} V = \operatorname{Dom} X^*$ and $\Gamma(V) \subset \Gamma(QUQU)$ because $QUQU$ is a unitary transformation defined on all of H.

THEOREM 2. *If X is a closed, densely defined linear involution which is conjugate symmetric, then X is conjugate selfadjoint.*

PROOF. Let K be the linear transformation given by $K = 2|X| + Q|X|^{-1}Q$. According to Lemma 2, $\operatorname{Dom} K = \operatorname{Dom}|X|$, and we can thus write $K = (2 + Q|X|^{-1}Q|X|^{-1})|X|$. An application of Lemma 3 now gives $K = R|X|$, where $R = 2 + QUQU$. Since R is a bounded linear transformation on H whose spectrum $\sigma(R)$ has the property that $\sigma(R) \subset \{z|\operatorname{Re} z \geqslant 1\}$, it follows that K is a closed linear operator with $\operatorname{Dom} K = \operatorname{Dom}|X|$. Moreover, since $\Gamma(Y^* + Z^*) \subset \Gamma((Y + Z)^*)$ for densely defined linear transformations [7, p. 300], we obtain $\Gamma(K) \subset \Gamma(K^*)$. Hence $\Gamma(R|X|) \subset \Gamma(|X|R^*)$ and it follows that $(R^* \oplus R)\Gamma(|X|) = \Gamma(R|X|R^{*-1}) \subset \Gamma(|X|)$. Since $\sigma(R^* \oplus R) = \sigma(R^*) \cup \sigma(R) \subset \{z|\operatorname{Re} z \geqslant 1\}$, any invariant subspace for $R^* \oplus R$ is also an invariant subspace for $(R^* \oplus R)^{-1}$ [**6**, p. 33]. Consequently $(R^* \oplus R)\Gamma(|X|) = \Gamma(|X|)$, and we conclude that $QUQU|X| = |X|UQUQ$. This last equation implies

$$UQUQ(\operatorname{Dom} X) = \operatorname{Dom} X = QUQU(\operatorname{Dom} X),$$

so we finally have $X = QX^*Q$.

COROLLARY. *If X is a conjugate symmetric linear involution and Y is a closed densely defined linear operator having the property that $\Gamma(Y) \subset \Gamma(X)$, then $Y = X$.*

PROOF. From the hypotheses we at once find

$$\Gamma(Y) \subset \Gamma(X) \subset \Gamma(QX^*Q) \subset \Gamma(QY^*Q).$$

Consequently $QY^*Q = Y$ by virtue of Theorem 2, and this implies $X = Y$.

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