BEST MONOTONE APPROXIMATION IN $L_1[0, 1]$

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Abstract. If $f$ is a bounded Lebesgue measurable function on $[0, 1]$ and $1 < p < \infty$, let $f_p$ denote the best $L_p$-approximation to $f$ by nondecreasing functions. It is shown that $f_p$ converges almost everywhere as $p$ decreases to one to a best $L_1$-approximation to $f$ by nondecreasing functions. The set of best $L_1$-approximations to $f$ by nondecreasing functions is shown to include its supremum and infimum.

Let $\Omega = [0, 1]$, $\mu =$ Lebesgue measure and $\mathcal{A} =$ the Lebesgue measurable subsets of $\Omega$. For $1 \leq p \leq \infty$, let $L_p = L_p(\Omega, \mathcal{A}, \mu)$. Let $M$ denote the set of all nondecreasing functions on $\Omega$. Suppose $f \in L_\infty$. For $1 < p < \infty$, $L_p$ is a uniformly convex Banach space and $M$ is a closed convex subset thereof, so $f$ has a unique best $L_p$-approximation $f_p$ by elements of $M$, i.e., $f_p$ is the unique element of $M$ which satisfies

$$\|f - f_p\|_p = \inf\{\|f - h\|_p: h \in M\}.$$ 

The function $f$ is said to have the Polya property if $\lim_{p \to \infty} f_p$ exists almost everywhere as a bounded measurable function and the Polya-one property if $\lim_{p \to 1} f_p$ exists in the same way. The Polya property fails for an arbitrary $f$ in $L_\infty [2, 1]$ but, as is shown in this note, the Polya-one property obtains.

If $B$ is a subset of $L_1$, let $\mu_1(f|B)$ denote the set of all best $L_1$-approximations of $f$ in $B$ and let $f(B) = \inf\mu_1(f|B)$, $\tilde{f}(B) = \sup\mu_1(f|B)$. If $\mathcal{B}$ is a subsigma algebra of $\mathcal{A}$ and $B$ is the subspace of $L_1$ consisting of all $\mathcal{B}$-measurable functions, then $f(B)$ and $\tilde{f}(B)$ are in $\mu_1(f|B)$ and $g \in \mu_1(f|B)$ if and only if $f(B) \leq g \leq \tilde{f}(B)$ [5]. Let $f = f(M)$ and $\tilde{f} = \tilde{f}(M)$. In this note we show that $f, \tilde{f} \in \mu_1(f|M)$ and that every convex combination of $f$ and $\tilde{f}$ is in $\mu_1(f|M)$ (so that $f$ and $\tilde{f}$ are extreme points of the $L_1$-compact convex set $\mu_1(f|M)$), but there may be a function $g \in M$ such that $f \leq g \leq \tilde{f}$ but $g$ is not in $\mu_1(f|M)$.

**Lemma 1.** $M$ is an $L_1$-closed convex subset of $L_1$, and $\mu_1(f|M)$ is a nonempty subset of $L_\infty$.

**Proof.** Suppose $\{g_n: n = 1, 2, \ldots \} \subset M$ and $g_n \to g$ in $L_1$. Since $\{g_n\}$ has a subsequence which converges to $g$ almost everywhere, we may assume that $g_n \to g$ almost everywhere. Let $\tilde{g} = \limsup_{n \to \infty} g_n$. Since each $g_n$ is nondecreasing, $\tilde{g}$ is nondecreasing. Thus $g$ is equivalent to an element of $M$. Clearly $M$ is convex.
Lemma 4 in [1] shows that \( \mu_1(f|M) \) is nonempty. If \( g \in \mu_1(f|M) \), it is clear that \( \|g\|_\infty \leq \|f\|_\infty \) so \( \mu_1(f|M) \subset L_\infty \). This establishes Lemma 1.

The next theorem shows that every bounded measurable function has the Polya-one property when \( M \) is the set from which best approximations are chosen. Let \( f_1 = m_1(f|M) \), the unique element of \( \mu_1(f|M) \) which minimizes

\[
\left\{ \int |f - h| \ln |f - h| : h \in \mu_1(f|M) \right\}.
\]

The function \( f_1 \) is termed by Landers and Rogge [4] the "natural" best \( L_1 \)-approximation.

**THEOREM 2.** If \( f \in L_\infty \), then \( f_p \) converges almost everywhere as \( p \) decreases to one to an element of \( \mu_1(f|M) \).

**PROOF.** We claim that \( f_p \to f_1 \) almost everywhere as \( p \downarrow 1 \). Suppose this is not the case. Then there exists a sequence \( \{p_n\} \) such that \( p_n \downarrow 1 \) and a set \( E \subset \Omega \) with \( \mu E > 0 \) and, for each \( x \) in \( E, f_{p_n}(x) \) does not converge to \( f_1(x) \).

Since \( f_1 \) is nondecreasing, the set of points of discontinuity of \( f_1 \) is at most countable. Thus there is a point \( y \) in \( \Omega \) at which \( f_1 \) is continuous but \( f_{p_n}(y) \) does not converge to \( f_1(y) \), whence there exists a subsequence \( \{q_n\} \) of \( \{p_n\} \) such that

\[
\lim_{n \to \infty} f_{q_n}(y) = d \neq f_1(y).
\]

By [4, Theorem 2], \( f_{q_n} \) converges strongly in \( L_1 \) to \( f_1 \). Thus, there exists a subsequence \( \{r_n\} \) of \( \{q_n\} \) such that \( f_{r_n} \to f_1 \) a.e. By Helly's Theorem [3, p. 221], there exist a nondecreasing function \( h \) and a subsequence \( \{s_n\} \) of \( \{r_n\} \) such that \( f_{s_n} \to h \) pointwise. Since \( f_{s_n} \to f_1 \) a.e., \( f_1 = h \) a.e. Since \( h(y) = d \), \( f_1 \) is continuous at \( y \) and \( h \) is nondecreasing, \( \mu[f_1 \neq h] > 0 \), a contradiction. This establishes Theorem 2.

For functions \( g, h : \Omega \to R \), let \( g \vee h \) be defined by \( g \vee h(x) = \max\{g(x), h(x)\} \). Replacing \( \max \) by \( \min \) defines \( g \wedge h \). Let \( C(g) \) denote the set of points of continuity of \( g \).

**LEMMA 3.** If \( g, h \in \mu_1(f|M) \), then \( g \vee h \) and \( g \wedge h \) are in \( \mu_1(f|M) \).

**PROOF.** Clearly \( g \vee h \) and \( g \wedge h \) are in \( M \). Our proof that they are also best \( L_1 \)-approximations of \( f \) will rely on the fact that each of the sets \([g > h]\) and \([g < h]\) is equivalent to an open set.

Let \( A = (0,1) \cap [g > h] \cap C(g) \cap C(h) \). Then \( \mu A = |g > h| \). For a given \( y \) in \( A \), let

\[
s = \begin{cases} \sup\{x < y : g(x) < h(x)\} & \text{if the set is nonempty,} \\ 0 & \text{otherwise,} \end{cases}
\]

and let

\[
t = \begin{cases} \inf\{x > y : g(x) < h(x)\} & \text{if the set is nonempty,} \\ 1 & \text{otherwise.} \end{cases}
\]

Then \( s < y < t \) and \( g > h \) on \((s, t)\). In any interval of the form \((t, z)\), there exists a point \( w \) such that \( h(w) \geq g(w) \) so, for \( x \geq w \), \( h(x) \geq h(w) \geq g(w) \geq g(t) \), whence

\[
\lim_{x \downarrow t} h(x) = g(t).
\]
Define \( \theta \in M \) by

\[
\theta(x) = \begin{cases} 
    h(x), & 0 \leq x \leq s, \\
    g(x), & s < x < t, \\
    \lim_{z \downarrow t} h(z), & x = t, \\
    h(x), & t < x \leq 1.
\end{cases}
\]

(1)

If \( \int_s^t |f - g| < \int_s^t |f - h| \), then

\[
\int_0^1 |f - \theta| = \int_0^s |f - h| + \int_s^t |f - g| + \int_t^1 |f - h| < \int_0^1 |f - h|,
\]

a contradiction. Thus \( \int_s^t |f - g| \geq \int_s^t |f - h| \). A similar argument shows that \( \int_s^t |f - g| \leq \int_s^t |f - h| \), and we see that \( \theta \in \mu_1(f|M) \).

Since \( y \) in \( A \) was arbitrary, the above arguments show that \( A \) is contained in a disjoint union of intervals \( \bigcup (s_i, t_i) \) such that \( g > h \) on \( (s_i, t_i) \) for each \( i \), and in each interval of the form \( (z, s_i) \) or \( (t_i, z) \) there exists a point \( w \) such that \( h(w) > g(w) \).

Define \( \theta_n \) in \( M \) by replacing \( s \) by \( s_n \) and \( t \) by \( t_n \) in (1) and, for \( n > 1 \), define \( \theta_n \) by

\[
\theta_n(x) = \begin{cases} 
    \theta_{n-1}(x), & 0 \leq x \leq s_n, \\
    g(x), & s_n < x < t_n, \\
    \lim_{z \downarrow t_n} \theta_{n-1}(z), & x = t_n, \\
    \theta_{n-1}(x), & t_n < x \leq 1.
\end{cases}
\]

Let \( \psi = \lim_{n \to \infty} \theta_n \). Then \( \psi \) is equivalent to \( g \lor h \) and, by the Dominated Convergence Theorem, \( \psi \in \mu_1(f|M) \). Thus \( g \lor h \in \mu_1(f|M) \).

The proof that \( g \land h \in \mu_1(f|M) \) is similar. This establishes Lemma 3.

If \( \{ g_n \} \subset \mu_1(f|M) \), then, by Helly’s Theorem, there is a subsequence \( \{ h_n \} \) of \( \{ g_n \} \) and there is a function \( h \in M \) such that \( h_n \to h \) pointwise. Since \( \{ h_n \} \) is uniformly bounded \( h_n \to h \) in \( L_1 \). Since \( h \in \mu_1(f|M) \), \( \mu_1(f|M) \) is \( L_1 \)-compact. A simple calculation shows that \( \mu_1(f|M) \) is convex. By the Krein-Milman Theorem, \( \mu_1(f|M) \) is the closed convex hull of its extreme points. The following theorem describes two of the extreme points of \( \mu_1(f|M) \).

**Theorem 4.** Each of the nondecreasing functions \( f \) and \( f \) is an element of \( \mu_1(f|M) \).

**Proof.** Let \( \{ r_i : i = 1, 2, \ldots \} \) be an enumeration of the rationals in \( \Omega \). Given \( i \), choose a sequence \( \{ g_n \} \subset \mu_1(f|M) \) such that

\[
\lim_{n \to \infty} g_n(r_i) = \sup \{ g(r_i) : g \in \mu_1(f|M) \}.
\]

By Helly’s Theorem, there exist a nondecreasing function \( g \) and a subsequence of \( \{ g_n \} \) which converges to \( g \) pointwise. By the Dominated Convergence Theorem, \( g^n \in \mu_1(f|M) \). Let \( h^n = g^1 \lor g^2 \lor \cdots \lor g^n \). Lemma 3 and induction show that \( h^n \in \mu_1(f|M) \). Again by Helly’s Theorem, there exist \( h \) and \( M \) and a subsequence of \( \{ h^n \} \) which converges to \( h \) pointwise. As above, \( h \in \mu_1(f|M) \).

We now claim that \( h = \sup \mu_1(f|M) \) almost everywhere. Indeed, if \( x \) is rational, clearly \( h(x) = \sup \{ g(x) : g \in \mu_1(f|M) \} \). Suppose that \( x \in C(h) \) but \( h(x) < \sup \{ g(x) : g \in \mu_1(f|M) \} \). Then there exists a function \( g_0 \in \mu_1(f|M) \) such that
$h(x) < g_0(x)$. Since $x$ is in $C(h)$ and $g_0$ is in $M$, there exists an interval $I$ of the form $(y, x)$ or $(x, z)$ such that $h \neq \sup \mu_1(f|M)$ on $I$. Since $I$ contains a rational, this is impossible. Thus $h = \sup \mu_1(f|M)$ on $C(h)$. But $\mu C(h) = 1$.

The proof that $f \in \mu_1(f|M)$ is similar. This establishes Theorem 4.

We conclude with two examples. Let $f = I_{[0, 1/2]}$, the indicator function of $[0, 1/2]$. Then $f = 1$, $g = 0$, and $g(x) = x$ satisfies $f \leq g \leq \hat{f}$ but $\int_0^1 |f - g| > \int_0^1 |f - \hat{f}|$. Thus $g$ is not in $\mu_1(f|M)$, so the conjecture that the result of Shintani and Ando mentioned above extends to the case where $\mathcal{B}$ is any subsigma lattice is shown to be false.

Another possible conjecture is that $\mu_1(f|M)$ is exactly the set of all convex combinations of $f$ and $\hat{f}$, i.e., $f$ and $\hat{f}$ constitute the set of extreme points of $\mu_1(f|M)$. This conjecture also fails: let $f = I_{[0, 1/4]} + 3I_{[1/2, 3/4]} + 2I_{[3/4, 1]}$. Then the function $g = I_{[0, 1/2]} + 2I_{[1/2, 1]}$ is in $\mu_1(f|M)$ but is not a convex combination of $f$ and $\hat{f}$. Thus, a problem that remains open is to characterize the set of extreme points of $\mu_1(f|M)$.

**References**


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