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BEST MONOTONE APPROXIMATION IN $L_1[0,1]$

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ABSTRACT. If f is a bounded Lebesgue measurable function on $[0,1]$ and $1 < p < \infty$, let f_p denote the best L_p-approximation to f by nondecreasing functions. It is shown that f_p converges almost everywhere as p decreases to one to a best L_1-approximation to f by nondecreasing functions. The set of best L_1-approximations to f by nondecreasing functions is shown to include its supremum and infimum.

Let $\Omega = [0,1]$, $\mu =$ Lebesgue measure and $\mathfrak{A} =$ the Lebesgue measurable subsets of Ω. For $1 \leqslant p \leqslant \infty$, let $L_p = L_p(\Omega, \mathfrak{A}, \mu)$. Let M denote the set of all nondecreasing functions on Ω. Suppose $f \in L_\infty$. For $1 < p < \infty$, L_p is a uniformly convex Banach space and M is a closed convex subset thereof, so f has a unique best L_p-approximation f_p by elements of M, i.e., f_p is the unique element of M which satisfies

$$\|f - f_p\|_p = \inf\{\|f - h\|_p : h \in M\}.$$

The function f is said to have the *Polya property* if $\lim_{p\to\infty} f_p$ exists almost everywhere as a bounded measurable function and the *Polya-one* property if $\lim_{p\downarrow 1} f_p$ exists in the same way. The Polya property fails for an arbitrary f in L_∞ [**2, 1**] but, as is shown in this note, the Polya-one property obtains.

If B is a subset of L_1, let $\mu_1(f|B)$ denote the set of all best L_1-approximations of f in B and let $\underline{f}(B) = \inf \mu_1(f|B)$, $\bar{f}(B) = \sup \mu_1(f|B)$. If $\mathfrak{B}$ is a subsigma algebra of $\mathfrak{A}$ and B is the subspace of L_1 consisting of all $\mathscr{B}$-measurable functions, then $\underline{f}(B)$ and $\bar{f}(B)$ are in $\mu_1(f|B)$ and $g \in \mu_1(f|B)$ if and only if $\underline{f}(B) \leqslant g \leqslant \bar{f}(B)$ [**5**]. Let $\underline{f} = \underline{f}(M)$ and $\bar{f} = \bar{f}(M)$. In this note we show that $\underline{f}, \bar{f} \in \mu_1(f|M)$ and that every convex combination of $\underline{f}$ and $\bar{f}$ is in $\mu_1(f|M)$ (so that $\underline{f}$ and $\bar{f}$ are extreme points of the L_1-compact convex set $\mu_1(f|M)$), but there may be a function $g \in M$ such that $\underline{f} \leqslant g \leqslant \bar{f}$ but g is not in $\mu_1(f|M)$.

LEMMA 1. *M is an L_1-closed convex subset of L_1, and $\mu_1(f|M)$ is a nonempty subset of L_∞.*

PROOF. Suppose $\{g_n : n = 1, 2, \ldots\} \subset M$ and $g_n \to g$ in L_1. Since $\{g_n\}$ has a subsequence which converges to g almost everywhere, we may assume that $g_n \to g$ almost everywhere. Let $\bar{g} = \limsup_{n\to\infty} g_n$. Since each g_n is nondecreasing, $\bar{g}$ is nondecreasing. Thus g is equivalent to an element of M. Clearly M is convex.

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Lemma 4 in [1] shows that $\mu_1(f|M)$ is nonempty. If $g \in \mu_1(f|M)$, it is clear that $\|g\|_\infty \leqslant \|f\|_\infty$ so $\mu_1(f|M) \subset L_\infty$. This establishes Lemma 1.

The next theorem shows that every bounded measurable function has the Polya-one property when M is the set from which best approximations are chosen. Let $f_1 = m_1(f|M)$, the unique element of $\mu_1(f|M)$ which minimizes

$$\left\{ \int |f-h| \ln |f-h| : h \in \mu_1(f|M) \right\}.$$

The function f_1 is termed by Landers and Rogge [4] the "natural" best L_1-approximation.

THEOREM 2. *If $f \in L_\infty$, then f_p converges almost everywhere as p decreases to one to an element of $\mu_1(f|M)$.*

PROOF. We claim that $f_p \to f_1$ almost everywhere as $p \downarrow 1$. Suppose this is not the case. Then there exists a sequence $\{p_n\}$ such that $p_n \downarrow 1$ and a set $E \subset \Omega$ with $\mu E > 0$ and, for each x in E, $f_{p_n}(x)$ does not converge to $f_1(x)$.

Since f_1 is nondecreasing, the set of points of discontinuity of f_1 is at most countable. Thus there is a point y in Ω at which f_1 is continuous but $f_{p_n}(y)$ does not converge to $f_1(y)$, whence there exists a subsequence $\{q_n\}$ of $\{p_n\}$ such that $\lim_{n\to\infty} f_{q_n}(y) = d \neq f_1(y)$.

By [4, Theorem 2], f_{q_n} converges strongly in L_1 to f_1. Thus, there exists a subsequence $\{r_n\}$ of $\{q_n\}$ such that $f_{r_n} \to f_1$ a.e. By Helly's Theorem [3, p. 221], there exist a nondecreasing function h and a subsequence $\{s_n\}$ of $\{r_n\}$ such that $f_{s_n} \to h$ pointwise. Since $f_{s_n} \to f_1$ a.e., $f_1 = h$ a.e. Since $h(y) = d$, f_1 is continuous at y and h is nondecreasing, $\mu[f_1 \neq h] > 0$, a contradiction. This establishes Theorem 2.

For functions $g, h \colon \Omega \to R$, let $g \vee h$ be defined by $g \vee h(x) = \max\{g(x), h(x)\}$. Replacing max by min defines $g \wedge h$. Let $C(g)$ denote the set of points of continuity of g.

LEMMA 3. *If $g, h \in \mu_1(f|M)$, then $g \vee h$ and $g \wedge h$ are in $\mu_1(f|M)$.*

PROOF. Clearly $g \vee h$ and $g \wedge h$ are in M. Our proof that they are also best L_1-approximations of f will rely on the fact that each of the sets $[g > h]$ and $[g < h]$ is equivalent to an open set.

Let $A = (0,1) \cap [g > h] \cap C(g) \cap C(h)$. Then $\mu A = \mu[g > h]$. For a given y in A, let

$$s = \begin{cases} \sup\{x < y : g(x) \leqslant h(x)\} & \text{if the set is nonempty,} \\ 0 & \text{otherwise,} \end{cases}$$

and let

$$t = \begin{cases} \inf\{x > y : g(x) \leqslant h(x)\} & \text{if the set is nonempty,} \\ 1 & \text{otherwise.} \end{cases}$$

Then $s < y < t$ and $g > h$ on (s, t). In any interval of the form (t, z), there exists a point w such that $h(w) \geqslant g(w)$ so, for $x \geqslant w$, $h(x) \geqslant h(w) \geqslant g(w) \geqslant g(t)$, whence $\lim_{x \downarrow t} h(x) \geqslant g(t)$.

Define $\theta \in M$ by

$$\theta(x) = \begin{cases} h(x), & 0 \leqslant x \leqslant s, \\ g(x), & s < x < t, \\ \lim_{z \downarrow t} h(z), & x = t, \\ h(x), & t < x \leqslant 1. \end{cases} \tag{1}$$

If $\int_s^t |f - g| < \int_s^t |f - h|$, then

$$\int_0^1 |f - \theta| = \int_0^s |f - h| + \int_s^t |f - g| + \int_t^1 |f - h| < \int_0^1 |f - h|,$$

a contradiction. Thus $\int_s^t |f - g| \geqslant \int_s^t |f - h|$. A similar argument shows that $\int_s^t |f - g| \leqslant \int_s^t |f - h|$, so $\int_0^1 |f - \theta| = \int_0^1 |f - h|$ and we see that $\theta \in \mu_1(f|M)$.

Since y in A was arbitrary, the above arguments show that A is contained in a disjoint union of intervals $\bigcup (s_i, t_i)$ such that $g > h$ on (s_i, t_i) for each i, and in each interval of the form (z, s_i) or (t_i, z) there exists a point w such that $h(w) \geqslant g(w)$. Define θ_1 in M by replacing s by s_1 and t by t_1 in (1) and, for $n > 1$, define θ_n by

$$\theta_n(x) = \begin{cases} \theta_{n-1}(x), & 0 \leqslant x \leqslant s_n, \\ g(x), & s_n < x < t_n, \\ \lim\limits_{z \downarrow t_n} \theta_{n-1}(z), & x = t_n, \\ \theta_{n-1}(x), & t_n < x \leqslant 1. \end{cases}$$

Let $\psi = \lim_{n \to \infty} \theta_n$. Then ψ is equivalent to $g \vee h$ and, by the Dominated Convergence Theorem, $\psi \in \mu_1(f|M)$. Thus $g \vee h \in \mu_1(f|M)$.

The proof that $g \wedge h \in \mu_1(f|M)$ is similar. This establishes Lemma 3.

If $\{g_n\} \subset \mu_1(f|M)$, then, by Helly's Theorem, there is a subsequence $\{h_n\}$ of $\{g_n\}$ and there is a function $h \in M$ such that $h_n \to h$ pointwise. Since $\{h_n\}$ is uniformly bounded $h_n \to h$ in L_1. Since $h \in \mu_1(f|M), \mu_1(f|M)$ is L_1-compact. A simple calculation shows that $\mu_1(f|M)$ is convex. By the Krein-Milman Theorem, $\mu_1(f|M)$ is the closed convex hull of its extreme points. The following theorem describes two of the extreme points of $\mu_1(f|M)$.

THEOREM 4. *Each of the nondecreasing functions* $\underline{f}$ *and* $\bar{f}$ *is an element of* $\mu_1(f|M)$.

PROOF. Let $\{r_i: i = 1, 2, \ldots\}$ be an enumeration of the rationals in Ω. Given i, choose a sequence $\{g_n\} \subset \mu_1(f|M)$ such that

$$\lim_{n \to \infty} g_n(r_i) = \sup\{g(r_i): g \in \mu_1(f|M)\}.$$

By Helly's Theorem, there exist a nondecreasing function g^i and a subsequence of $\{g_n\}$ which converges to g^i pointwise. By the Dominated Convergence Theorem, $g^i \in \mu_1(f|M)$. Let $h^n = g^1 \vee g^2 \vee \cdots \vee g^n$. Lemma 3 and induction show that $h^n \in \mu_1(f|M)$. Again by Helly's Theorem, there exist h and M and a subsequence of $\{h^n\}$ which converges to h pointwise. As above, $h \in \mu_1(f|M)$.

We now claim that $h = \sup \mu_1(f|M)$ almost everywhere. Indeed, if x is rational, clearly $h(x) = \sup\{g(x): g \in \mu_1(f|M)\}$. Suppose that $x \in C(h)$ but $h(x) < \sup\{g(x): g \in \mu_1(f|M)\}$. Then there exists a function g_0 in $\mu_1(f|M)$ such that

$h(x) < g_0(x)$. Since x is in $C(h)$ and g_0 is in M, there exists an interval I of the form (y, x) or (x, z) such that $h \neq \sup \mu_1(f|M)$ on I. Since I contains a rational, this is impossible. Thus $h = \sup \mu_1(f|M)$ on $C(h)$. But $\mu C(h) = 1$.

The proof that $\underline{f} \in \mu_1(f|M)$ is similar. This establishes Theorem 4.

We conclude with two examples. Let $f = I_{[0,1/2]}$, the indicator function of $[0, 1/2]$. Then $\bar{f} \equiv 1$, $\underline{f} \equiv 0$, and $g(x) = x$ satisfies $\underline{f} \leqslant g \leqslant \bar{f}$ but $\int_0^1|f - g| > \int_0^1|f - \underline{f}|$. Thus g is not in $\mu_1(f|M)$, so the conjecture that the result of Shintani and Ando mentioned above extends to the case where $\mathfrak{B}$ is any subsigma lattice is shown to be false.

Another possible conjecture is that $\mu_1(f|M)$ is exactly the set of all convex combinations of $\underline{f}$ and $\bar{f}$, i.e., $\underline{f}$ and $\bar{f}$ constitute the set of extreme points of $\mu_1(f|M)$. This conjecture also fails: let $f = I_{[0,1/4]} + 3I_{(1/2,3/4]} + 2I_{(3/4,1]}$. Then the function $g = I_{[0,1/2]} + 2I_{(1/2,1]}$ is in $\mu_1(f|M)$ but is not a convex combination of $\underline{f}$ and $\bar{f}$. Thus, a problem that remains open is to characterize the set of extreme points of $\mu_1(f|M)$.

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