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VISCOSITY SOLUTIONS OF HAMILTON-JACOBI EQUATIONS AT THE BOUNDARY[1]

MICHAEL G. CRANDALL AND RICHARD NEWCOMB

ABSTRACT. When considering classical solutions of boundary value problems for nonlinear first-order scalar partial differential equations, one knows that there are parts of the boundary of the region under consideration where one cannot specify data and would not expect to require data in order to prove uniqueness. Of course, classical solutions of such problems rarely exist in the large owing to the crossing of characteristics. The theory of a sort of generalized solution—called "viscosity solutions"—for which good existence and uniqueness theorems are valid has been developed over the last few years. In this note we give some results concerning parts of the boundary on which one need not know (prescribe) viscosity solutions to be able to prove comparison (and hence uniqueness) results. In this context, this amounts to identifying boundary points with the property that solutions in the interior which are continuous up to the boundary are also viscosity solutions at the boundary point. Examples indicating the sharpness of the results are given.

0. Introduction. We begin by recalling an important notion of generalized solutions for scalar nonlinear first order partial differential equations. Let K be a subset of $\mathbf{R}^M$, and let F: $K \times \mathbf{R} \times \mathbf{R}^M \to \mathbf{R}$ be continuous (i.e., $F \in C(K \times \mathbf{R} \times \mathbf{R}^M)$). A function $u \in C(K)$ is called a viscosity solution of $F(y, u, Du) \leqslant 0$ on K if, for each real-valued function φ which is continuously differentiable in a neighborhood of K and each local maximum $z \in K$ of $u - \varphi$ relative to K, one has

$$F(z, u(z), D\varphi(z)) \leqslant 0. \tag{0.1}$$

Here $D\varphi = (\varphi_{y_1}, \ldots, \varphi_{y_M})$ is the gradient of φ. We will use the notation $C^1(K)$ to mean the set of functions which are defined and continuously differentiable in a neighborhood of K. Similarly, a viscosity solution of $F(y, u, Du) \geqslant 0$ in K is a $u \in C(K)$ such that, for every $\varphi \in C^1(K)$ and local minimum $z \in K$ of $u - \varphi$ relative to K, one has

$$F(z, u(z), D\varphi(z)) \geqslant 0. \tag{0.2}$$

A viscosity solution of $F = 0$ on K is a function which is a viscosity solution of both $F \leqslant 0$ and $F \geqslant 0$. We also call viscosity solutions of $F \leqslant 0$ ($F \geqslant 0$) viscosity subsolutions (respectively, supersolutions) of $F = 0$. Observe that these notions do

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not require u to be anywhere differentiable. Indeed, although we will not do so here, there are circumstances when it is appropriate to require only a semicontinuity of u rather than continuity. The set K is also general—we have not yet restricted it in any way—but we will primarily be concerned with cases in which K satisfies $\Omega \subset K \subset \overline{\Omega}$, where Ω denotes an open subset of $\mathbf{R}^M$, $\overline{\Omega}$ is its closure, and $\partial\Omega$ is its boundary.

The notion of viscosity solutions has become important in providing a theoretical basis for the interaction between equations of Hamilton-Jacobi type, control theory, and differential games. The first uniqueness theorems for this notion are proved in Crandall and Lions [**4**]. However, a more complex equivalent formulation is taken as basic in [**4**], and Crandall, Evans, and Lions [**3**] give direct proofs with the simpler formulation. Lions [**8**] and Crandall and Souganidis [**6**] provide a view of the scope of the theory and the references to much of the recent literature.

While one primarily had in mind the case in which K is open in the theory referred to earlier, [**4**, Remark 1.13] pertained to the general case. Moreover, in [**4**, Proposition V.1] it is proved that a viscosity solution of

$$u_t + H(x, t, u, Du) = 0 \quad \text{on } \mathbf{O} \times (0, T), \tag{0.3}$$

where $\mathbf{O}$ is an open subset of $\mathbf{R}^N$ and $Du = (u_{x_1}, \ldots, u_{x_N})$ is the spatial gradient of u, which happens to extend continuously to $\mathbf{O} \times (0, T]$, is also a viscosity solution on the set $\mathbf{O} \times (0, T]$. See also [**3**, Lemma 4.1]. Of course, (0.3) is subsumed under the general case by putting $M = N + 1, y_1 = x_1, \ldots, y_{M-1} = x_N$, and $y_M = t$, and

$$F(y, u, (p_1, \ldots, p_{N+1})) = p_{N+1} + H((y_1, \ldots, y_N), y_{N+1}, u, (p_1, \ldots, p_N)).$$

It is further remarked in [**4**] that this extension property depends on a certain monotonicity of the equation in the direction of the normal to the domain, and this is the point we examine in some generality in this note.

Our current interest in this point is partly generated by recent remarks of R. Jensen [**7**], who had the idea of formulating the uniqueness theorem on arbitrary closed sets, a formulation with attractive features. Our main results, which we introduce and prove in §1, are criteria which identify boundary points at which an inequation is automatically satisfied in the viscosity sense if it holds in the interior of a set, extending the result for (0.3). Examples establishing the sharpness of the result are also given, and we formulate corresponding uniqueness theorems. We would also like to mention the recent paper [**10**] by H. Soner, which the reader may find of interest.

1. The extension theorem and examples. In this section Ω denotes an open subset of $\mathbf{R}^M$ and $\partial\Omega$ is its boundary. We consider sub- and supersolutions $u \in C(\overline{\Omega})$ of an equation $F = 0$ on Ω and define a subset I_F of $\partial\Omega$ (which we call the part of $\partial\Omega$ irrelevant for F). In nice situations u is then a sub- or supersolution (as appropriate) of $F = 0$ on $\Omega \cup I_F$. We are concerned about how general u, Ω and F may be and still have the result hold. With this in mind, we make the definitions for a general open set Ω. We need to define an appropriate set of normals to a point $z \in \partial\Omega$. The open ball of radius r centered at z in $\mathbf{R}^M$ will be denoted by $B_r(z)$, i.e.,

$$B_r(z) = \{ y \in \mathbf{R}^M \colon |y - z| < r \}.$$

DEFINITION 1. Let $z \in \partial\Omega$. Then $\nu \in \mathbf{R}^M \setminus \{0\}$ is an inward normal to Ω at z if there is a $\lambda > 0$ for which the open ball of radius $|\lambda\nu|$ centered at $z + \lambda\nu$ is contained in Ω, i.e.,

$$B_{|\lambda\nu|}(z + \lambda\nu) \subset \Omega.$$

The set of all inward normals to Ω at $z \in \partial\Omega$ is denoted by $N_\Omega(z)$. Another description of $N_\Omega(z)$ can be given the following way: For each $y \in \mathbf{R}^M$ there are points of $\partial\Omega$ which are nearest y. Let Py be the set of such points:

$$Py = \{ z \in \partial\Omega \colon |y - z| \leqslant |y - w| \text{ for } w \in \partial\Omega \}. \tag{1.1}$$

Then

$$N_\Omega(z) = \{ \lambda(y - z) \colon y \in \Omega, z \in Py, \lambda > 0 \}, \tag{1.2}$$

that is, the inward normal vectors are just those in the directions from z to points in Ω for which z is a nearest point in $\partial\Omega$. We remark that there are a variety of choices for the definitions of "normal" and "tangent" vectors to an arbitrary set. We are using a notion appropriate for our purposes—as regards others and relationships among them, see Clarke [**2**] and Aubin and Ekeland [**1**]. The reader can easily convince himself that $N_\Omega(z)$ may be empty, and it may be $\mathbf{R}^M \setminus \{0\}$ when $z \in \partial\Omega$.

DEFINITION 2. Let $z \in \partial\Omega$. Then $z \in I_F$ if there is an $r > 0$ such that for all $y \in \partial\Omega \cap B_r(z)$, all $\nu \in N_\Omega(y)$, and all $(u, p) \in \mathbf{R} \times \mathbf{R}^M$,

$$F(y, u, p + \nu) \leqslant F(y, u, p). \tag{1.3}$$

In other words, z is irrelevant for F if F is nonincreasing in the inward normal directions in a neighborhood of z. Of course, I_F depends on Ω as well as F, but we will not need to indicate this in this note.

REMARK. Observe that the set of vectors ν for which (1.3) holds for all $p \in \mathbf{R}^M$ is closed under addition. Using this and the continuity of F we deduce that if $z \in I_F$, then $F(z, u, p + \nu) \leqslant F(z, u, p)$ for all ν in the closed convex hull of $\limsup_{y \to z} N_\Omega(y)$, which may well be a much larger set than $N_\Omega(z)$.

We would like to prove that if $u \in C(\overline{\Omega})$ is a viscosity sub- or supersolution of $F = 0$ in Ω, then it is also a viscosity sub- or supersolution in $\Omega \cup I_F$, but this is not true without further restrictions. To formulate our first result we still need to define "regular points" of $\partial\Omega$. These will be defined by properties of the function

$$d(y) = \inf\{ |y - w|^2 \colon w \in \partial\Omega \}, \tag{1.4}$$

which is the square of the distance from y to $\partial\Omega$. Associated with $d(y)$ is the mapping P of (1.1). Indeed,

$$Py = \{ w \in \partial\Omega \colon d(y) = |y - w|^2 \}.$$

If Py is a singleton (i.e., there is only one point in $\partial\Omega$ nearest y), we will abuse notation and also use Py to denote this closest point.

DEFINITION 3. Let $z \in \partial\Omega$. Then z is regular for Ω if there is an $r > 0$ such that if $y \in B_r(z) \cap \Omega$, then Py is a singleton, d is differentiable at y, and

$$Dd(y) = 2(y - Py). \tag{1.5}$$

REMARK. It is a standard and elementary exercise to show that if Ω is of class C^2 near $z \in \partial\Omega$, then z is regular for Ω. In fact, d is differentiable at y and (1.5) holds exactly when Py is a singleton (e.g. [**2**]).

THEOREM 1. *Let $F \in C(\overline{\Omega} \times \mathbf{R} \times \mathbf{R}^M)$. Let $z \in I_F$ be regular for Ω, $u \in C(\Omega \cup I_F)$ be Lipschitz continuous near z, and u be a viscosity solution of $F \leqslant 0$ ($F \geqslant 0$) on Ω. Then u is also a viscosity solution of $F \leqslant 0$ (respectively, $F \geqslant 0$) on $\Omega \cup \{z\}$. In particular, if u is a viscosity solution of $F = 0$ on Ω, then it is a viscosity solution on $\Omega \cup \{z\}$.*

Before proving this result, we give two examples. The first shows that the restriction to Lipschitz continuous u's is necessary in this generality, while the second shows that if z is not regular for Ω the result may fail.

EXAMPLE 1. Consider the situation $\Omega = (0,1)$ and

$$F(x, u, u') = x^{\alpha}u' - 1 = 0,$$

where $0 < \alpha < 1$. In this example, $0 \in I_F$, since $F(0, u, p)$ is independent of p and hence is nonincreasing in all directions. Moreover, 0 is clearly regular for Ω. The function $u = x^{(1-\alpha)}/(1-\alpha)$ is a classical (and hence viscosity) solution of $F = 0$ on Ω. However, u is not Lipschitz continuous near 0. If $\varphi(x) = x$, then $u - \varphi$ has a minimum relative to $\overline{\Omega} = [0,1]$ at 0 but $F(0, u(0), \varphi'(0)) = -1 < 0$, so u is not a viscosity supersolution on $[0,1)$.

EXAMPLE 2. Let $M = 2$ and (x, y) denote points of $\mathbf{R}^2$. Put

$$\Omega = \{(x, y) \in \mathbf{R}^2 \colon 0 < y < x^2, 0 < x < 1\}$$

and

$$F(x, y, u, u_x, u_y) = u_x + 3(\sqrt{y})u_y. \tag{1.6}$$

We have

$$N_{\Omega}((x, y)) = \begin{cases} \varnothing & \text{if } (x, y) = (0,0), \\ \{\lambda(2x, -1) \colon \lambda > 0\} & \text{if } (x, y) = (x, x^2), 0 < x < 1, \\ \{(0, \lambda) \colon \lambda > 0\} & \text{if } (x, y) = (x, 0), 0 < x < 1, \end{cases}$$

and it is straightforward to check that $(0,0) \in I_F$. For example, F is nonincreasing in a direction $\nu = (\nu_x, \nu_y)$ at a point (x, x^2) exactly when (using the linear form (1.6) of F)

$$\nu_x + 3(\sqrt{x^2})\nu_y = \nu_x + 3x\nu_y \leqslant 0.$$

For the normal $\nu = (2x, -1)$ this quantity is $2x - 3x < 0$. Now $u = 0$ is a Lipschitz continuous viscosity solution of $F = 0$ in Ω. The function $0 - x = -x$ has a maximum on $\overline{\Omega}$ at $(0,0)$, but $F(0, 0, (1, 0)) = 1 > 0$, so 0 is not a viscosity subsolution on $\Omega \cup (0,0)$. (We conclude that $(0,0)$ is not regular and the requirement of regularity cannot be relaxed.)

PROOF OF THEOREM 1. Let $u \in C(\Omega \cup I_F)$ be (locally) Lipschitz continuous and a viscosity solution of $F \leqslant 0$ in Ω. Let $z \in I_F$ be regular for Ω. Put $\Omega_z = \Omega \cup \{z\}$. If

$\varphi \in C^1(\Omega_z)$ and $u - \varphi$ has z as a local maximum relative to Ω_z (and hence relative to a neighborhood of z in $\overline{\Omega}$), we need to prove that

$$F(z, u(z), D\varphi(z)) \leqslant 0. \tag{1.7}$$

Without loss of generality we may assume that for each small $r > 0$,

$$u(y) - \varphi(y) < u(z) - \varphi(z) \quad \text{for } y \in \overline{\Omega} \text{ and } |y - z| = r, \tag{1.8}$$

that is, the maximum is strict. This is because φ can be perturbed to $\varphi(y) + |y - z|^2$, which makes z a strict maximum without affecting $D\varphi(z)$. Let $\varepsilon > 0$. We claim that

$$\Psi_\varepsilon(y) = u(y) - \varphi(y) - \varepsilon/d(y) \tag{1.9}$$

has a local maximum $y_\varepsilon \in \Omega$ relative to Ω satisfying

$$y_\varepsilon \to z \quad \text{as } \varepsilon \to 0. \tag{1.10}$$

Indeed, because $u - \varphi$ is continuous, for each small $r > 0$ we can find $w \in B_r(z) \cap \Omega$ such that (1.8) holds with w in place of z:

$$u(y) - \varphi(y) < u(w) - \varphi(w) \quad \text{for } y \in \overline{\Omega} \text{ and } |y - z| = r. \tag{1.11}$$

Now consider Ψ_ε in the set $B_r(z) \cap \Omega$. We will argue informally, as it is best if the reader convinces himself of the validity of what follows: Ψ_ε tends to $-\infty$ on $\partial\Omega$. On the other hand, we can guarantee that $\Psi_\varepsilon(y) < \Psi_\varepsilon(w)$ on $|y - z| = r$ away from $\partial\Omega$ by choosing ε small, because of (1.11), and we can make $\Psi_\varepsilon(w)$ as close as we please to $u(w) - \varphi(w)$. The existence of local maxima y_ε satisfying (1.10) follows, as does the fact that we can guarantee

$$\Psi_\varepsilon(y_\varepsilon) \to u(z) - \varphi(z). \tag{1.12}$$

Next let z_ε be the nearest point to y_ε in $\partial\Omega$: $z_\varepsilon = Py_\varepsilon$. Then, by the assumption that u is a viscosity subsolution in Ω, the regularity of z, and (1.5),

$$F\left(y_\varepsilon, u(y_\varepsilon), D\varphi(y_\varepsilon) - 2\varepsilon(y_\varepsilon - z_\varepsilon)/(d(y_\varepsilon))^2\right) \leqslant 0. \tag{1.13}$$

With future uses in mind, we put

$$p_\varepsilon = D\varphi(y_\varepsilon), \quad \nu_\varepsilon = (y_\varepsilon - z_\varepsilon), \quad \lambda_\varepsilon = 2\varepsilon/(d(y_\varepsilon))^2, \tag{1.14}$$

and write (1.13) as

(1.15)

$$F(y_\varepsilon, u(y_\varepsilon), p_\varepsilon - \lambda_\varepsilon\nu_\varepsilon) - F(z_\varepsilon, u(y_\varepsilon), p_\varepsilon - \lambda_\varepsilon\nu_\varepsilon) + F(z_\varepsilon, u(y_\varepsilon), p_\varepsilon - \lambda_\varepsilon\nu_\varepsilon) \leqslant 0.$$

We treat the various terms of this inequality: First, since u is Lipschitz continuous,

$$|p_\varepsilon - \lambda_\varepsilon\nu_\varepsilon| \leqslant L, \tag{1.16}$$

where L is a Lipschitz constant for u in the neighborhood of z in which y_ε lies. (See [**4**, Lemma II.3].) Since $y_\varepsilon, z_\varepsilon \to z$, we may use (1.16) and the continuity of F to conclude that the difference comprised by the first two terms in (1.15) tends to 0 with ε. As regards the last term, observe that $z_\varepsilon \to z$ and $\nu_\varepsilon \in N_\Omega(z_\varepsilon)$ imply, because $z \in I_F$, that

$$F(z_\varepsilon, u(y_\varepsilon), p_\varepsilon - \lambda_\varepsilon\nu_\varepsilon) \geqslant F(z_\varepsilon, u(y_\varepsilon), p_\varepsilon). \tag{1.17}$$

Using (1.17) and the prior remark together with (1.15), we may pass to the limit as $\varepsilon \to 0$ to find (1.7) as desired. The case of supersolutions is handled in a parallel way, and the case of solutions follows from the sub and super cases.

At this point we have not even recovered the results for (0.3) mentioned above, as they do not require Lipschitz continuity of u. By Example 1 it is clear that further restrictions are needed on F in order to deal with more general u's. Looking at the proof of Theorem 1 and Example 1 will make the condition we are about to formulate more palatable.

DEFINITION 4. Let $z \in I_F$. Then z is regular for F if for all sequences $z_n \in \partial\Omega$ and $y_n \in (N_\Omega(z_n) + z_n) \cap \Omega$ convergent to z and $\lambda_n > 0$ satisfying

$$\lambda_n |y_n - z_n|^2 \to 0 \tag{1.18}$$

we have

$$\liminf_{n \to \infty} F(y_n, u, p - \lambda_n(y_n - z_n)) - F(z_n, u, p - \lambda_n(y_n - z_n)) \geqslant 0 \tag{1.19}$$

uniformly for bounded u and p.

In particular, the continuity requirement we are imposing is laid only on the behaviour of F near z and in appropriate directions. We have

THEOREM 2. *Let $F \in C(\bar{\Omega} \times \mathbf{R} \times \mathbf{R}^M)$. Let $z \in I_F$ be regular for Ω and F, and $u \in C(\Omega \cup I_F)$ be a viscosity solution of $F \leqslant 0$ ($F \geqslant 0$) on Ω. Then u is also a viscosity solution of $F \leqslant 0$ (respectively, $F \geqslant 0$) on $\Omega \cup \{z\}$. In particular, if u is a viscosity solution of $F = 0$ on Ω, then it is a viscosity solution on $\Omega \cup \{z\}$.*

PROOF. The proof follows the proof of Theorem 1 exactly up to the discussion of (1.15). With the notation (1.14), the relation (1.16) no longer holds for any L. However, p_ε is bounded and if we observe that

$$\lambda_\varepsilon |\nu_\varepsilon|^2 = \lambda_\varepsilon |y_\varepsilon - z_\varepsilon|^2 = 2\varepsilon / d(y_\varepsilon) \to 0,$$

as follows immediately from (1.12) and the definition of Ψ_ε, then the assumption that z is regular for F may be used to claim that

$$\liminf_{\varepsilon \to 0} \left(F(y_\varepsilon, u(y_\varepsilon), p_\varepsilon - \lambda_\varepsilon \nu_\varepsilon) - F(z_\varepsilon, u(y_\varepsilon), p_\varepsilon - \lambda_\varepsilon \nu_\varepsilon) \right) \geqslant 0,$$

and the proof is completed as before.

EXAMPLE 3. We may use the situation of Example 1 to show that the "rate" in (1.18) is sharp among power laws. Indeed, with the notation of Example 1, if $z_n = 0$ for all n and y_n is a sequence of positive numbers convergent to zero which satisfies

$$\lambda_n |y_n - z_n|^{1+\alpha} = \lambda_n (y_n)^{1+\alpha} \to 0,$$

we have

$$F(y_n, u, p - \lambda_n y_n) - F(0, u, p - \lambda_n y_n) = y_n^\alpha p - \lambda_n y_n^{1+\alpha} \to 0$$

uniformly for bounded p. That is, the assumptions of Theorem 2 are satisfied except that the exponent 2 in (1.18) is replaced by $1 + \alpha$. Yet we know that the viscosity solution $x^{(1-\alpha)}/(1 - \alpha)$ of $F = 0$ on $(0,1)$ is not a supersolution on $[0, 1)$.

REMARKS. The proofs of Theorems 1 and 2 couple the general line of argument used in the special case (0.3) in [**4** and **3**] with the use of the distance function to replace the particular construction used in this case. Use of the distance function is frequently advantageous in proofs in this subject (see, for example, Lions [**8**, **9**] and Jensen [**7**]). By the way, it is an elementary (and standard) exercise to show that $u(x) = (d(x))^{1/2}$ is a viscosity solution of $|Du|^2 = 1$ in Ω whether or not $\partial\Omega$ is smooth. It is the only viscosity solution of this equation vanishing on $\partial\Omega$ (see [**8** and **4**]).

Finally, we formulate uniqueness theorems corresponding to Theorems 1 and 2. There are many possible variants of these results, and these are chosen to simply illustrate the interaction between Theorems 1 and 2 and uniqueness. Corresponding to Theorem 1 we have

THEOREM U1. *Let Ω be a bounded open subset of $\mathbf{R}^M$. Let $F \in C(\overline{\Omega} \times \mathbf{R} \times \mathbf{R}^M)$ satisfy*

> *for $R > 0$ there is a strictly increasing function γ_R such that $\gamma_R(0) = 0$ and $F(y, u, p) - F(y, v, p) \geqslant \gamma_R(u - v)$ for $y \in \overline{\Omega}$, $p \in \mathbf{R}^M$, and $u, v \in [-R, R]$.*

Let $u, v \in C(\overline{\Omega})$ be Lipschitz continuous, u be a viscosity solution of $F \leqslant 0$, and v be a viscosity solution of $F \geqslant 0$ on Ω. Let each point of I_F be regular for Ω. Let $u \leqslant v$ on $\partial\Omega \setminus I_F$, then $u \leqslant v$ in $\overline{\Omega}$.

Corresponding to Theorem 2 we have

THEOREM U2. *Let Ω and F satisfy the assumptions of Theorem* U1. *In addition, assume that each point of I_F is regular for Ω and F and that for each R there is a continuous function g_R: $[0, \infty) \to [0, \infty)$ with $g_R(0) = 0$ such that*

$$F(x, u, \lambda(x - y)) - F(y, u, \lambda(x - y)) \geqslant g_R\left(\lambda|x - y|^2 + |x - y|\right) \tag{1.20}$$

for $x, y \in \overline{\Omega}$, $\lambda > 0$, and $|u| \leqslant R$. Let $u, v \in C(\overline{\Omega})$, u be a viscosity solution of $F \leqslant 0$, v be a viscosity solution of $F \geqslant 0$, and $u \leqslant v$ on $\partial\Omega \setminus I_F$. Then $u \leqslant v$ in $\overline{\Omega}$.

REMARK. Condition (1.20) on F is a weakened version of a uniqueness condition used in [**4**]. The relevance of such a "one-sided" condition was pointed out by R. Jensen. See also [**5**] for uniqueness (in unbounded domains) and existence using this condition, as well as a generalization of it.

The proofs of Theorems U1 and U2, given Theorems 1 and 2, are routine and will not be given here. See, however, Jensen [**7**] concerning general formulations of results on closed sets.

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Mathematics Research Center, University of Wisconsin, Madison, Wisconsin 53705

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706