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SYMMETRIC CUT LOCI IN RIEMANNIAN MANIFOLDS

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ABSTRACT. Let M be a compact Riemannian manifold with $H_1(M, Z) = 0$. We show that, for a point $p \in M$, the cut locus and conjugate locus of p must intersect if M admits a group of isometries which fixes p and has principal orbits of codimension at most 2. This is a classical theorem of Myers [5] in the case when M has dimension 2.

0. In [5] Myers proved that if M is a Riemannian manifold homeomorphic to S^2 and $p \in M$, then the cut locus and conjugate locus of p in the tangent space M_p must have a common point (also see Theorem 5.1 of [10]). On the other hand, Weinstein [10], answering a problem of Rauch [7], constructed a Riemannian metric on any compact simply-connected C^∞ manifold not homemorphic to S^2, so that there is a point $p \in M$ whose conjugate and cut loci are disjoint. The following conjecture was proposed by Weinstein [10]: "If M is a compact simply-connected Riemannian manifold, then for some point $p \in M$, the conjugate locus and cut locus of p intersect." Gromov has recently constructed metrics on S^3 with sectional curvature $\leqslant 1$ and arbitrarily small diameter, thus disproving this conjecture.

We give the following extension of Myers' result.

THEOREM *Suppose M is a compact, connected, C^∞ Riemannian manifold and there is a compact Lie group G of isometries of M which fix some point $p \in M$. Assume that $H_1(M, Z) = 0$ and that a principal orbit of the G-action has codimension* 2. *Then the conjugate locus and cut locus of p must have a point in common.*

1. Remarks. (a) If M has dimension 2, then, since $H_1(M, Z) = 0$, it follows that M is homeomorphic to S^2. If we take G to be the trivial group, then the theorem becomes Myers' result.

(b) All the 3-dimensional lens spaces $L(m, n)$ (see e.g. [6]) with the standard spherical metric admit S^1-actions which fix points p. Also the cut and conjugate loci of p are disjoint, but $H_1(L(m, n), Z) = Z_m$.

(c) The Poincaré dodecahedral space M^3 (see [6]) with metric induced from S^3 is a homogeneous space admitting a transitive SU(2)-action. Moreover, $H_1(M, Z) = 0$ and the cut and conjugate loci of any point are disjoint. However, the isotropy subgroup of any point is finite, so it has principal orbits of codimension 3.

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(d) In Berger's classification [**1**] of normal Riemannian homogeneous spaces of strictly positive curvature, a class of Riemannian metrics on odd-dimensional spheres S^{2n+1}, of the form $\mathrm{SU}(n+1) \times \mathbf{R}/\mathrm{SU}(n) \times \mathbf{R}$, is given. It is easy to see that these examples satisfy the hypotheses of the theorem, and, hence, the conjugate and cut loci of any point must intersect. Note that the conjugate locus of a point in these manifolds is calculated in [**3**], and the cut locus, in the case $n = 1$, is computed in [**8**]. The result of the theorem applied to these examples of Berger for the case $n = 1$ is also given in [**9**].

2. Following Bredon [**2**], we introduce some transformation group notation. Let M be a compact C^∞ manifold, and let G be a compact Lie grouup acting smoothly on M. The orbits G_p, $p \in M$, are partially ordered by the relation $G_p \leqslant G_q$ if the isotropy subgroup of p is conjugate to a subgroup of the isotropy subgroup of q. A maximal orbit type is called a principal orbit, and the union of all principal orbits is labelled U.

The nonprincipal orbits are of two types. Let d be the dimension of a principal orbit. Orbits with dimension strictly less than d are called *singular*, while nonprincipal orbits with dimension d are called *exceptional*. The union in M of the singular (resp. exceptional) orbits is denoted by B (resp. E).

Let M^* denote the orbit space. If S is a G-invariant set in M, let S^* denote the projection of S to M^*. Then U (resp. U^*) is an open dense subset of M (resp M^*). (See [**2**, Theorem 3.1, p. 179].) If $\dim M = n$ and $d = n - 1$ or $n - 2$, then M^* is a manifold, possibly with boundary (cf. [**2**, Lemma 4.1, p. 186]).

With the notation of the theorem, let $C(p)$ (resp. $\tilde{C}(p)$) denote the cut locus of p in M (resp. M_p). Note that $\tilde{C}(p)$ is homeomorphic to S^{n-1}. The action of G on M can be lifted to a linear action of G on the tangent space M_p. We let $\tilde{U}$ (resp. $\tilde{B}$, $\tilde{E}$) denote the union of the principal (resp. singular, exceptional) orbits in M_p. Finally, let $\tilde{D}(p)$ be the cell which is the closure of the bounded component of $M_p - \tilde{C}(p)$.

3. Proof of the Theorem. If $\dim M = 2$, the result follows by Myers' theorem (cf. [**5** and **10**, Theorem 5.1]). Therefore we can assume $\dim M \geqslant 3$. Now the action of G on $\tilde{D}(p)$ can be regarded as the cone of the action of G on $\tilde{C}(p)$ with the origin as vertex, since $\tilde{D}(p)$ is star-like from the origin and the G-action on M_p is linear. By [**2**, Theorem 8.2, p. 206], $\tilde{C}(p)^*$ is homeomorphic to either S^1 or $[0,1]$, since the principal orbits for the G-action on $\tilde{C}(p)$ have codimension one. In the former case $\tilde{C}(p)^*$ is a bundle over S^1, which gives a contradiction (by the homotopy sequence of a fibration applied to the $(n-1)$-sphere $\tilde{C}(p)$). So $\tilde{D}(p)^*$ is a cone on the interval $\tilde{C}(p)^*$.

Since $\dim M \geqslant 3$, p is a singular orbit of the G-action on M, so $B^* \neq \varnothing$. Therefore, all the hypotheses of Theorem 8.6 in [**2**, p. 211] are satisfied for the G-action on M. We conclude that $E^* = \varnothing$, M^* is a 2-disk with boundary B^*, and $\operatorname{int} M^* = U^*$.

Let $\exp^*\colon M_p^* \to M^*$ be the map between orbit spaces induced by the G-equivariant map $\exp\colon M_p \to M$. Since $\exp\colon \tilde{D}(p) - \tilde{C}(p) \to M - C(p)$ is a diffeomorphism, it follows that $\exp^*\colon \tilde{D}(p)^* - \tilde{C}(p)^* \to M^* - C(p)^*$ is a homeomorphism.

Suppose that $\tilde{C}(p)$ has no conjugate points, i.e., exp is a local diffeomorphism at each point of $\tilde{D}(p)$. Hence, G-orbit dimensions are preserved by exp restricted to $\tilde{D}(p)$, and exp* maps principal (resp. singular) orbits in $\tilde{D}(p)^*$ to principal (resp. singular) orbits in M^*. Furthermore, there are no exceptional orbits in $\tilde{D}(p)^*$ since E^* is empty.

exp*: $\tilde{D}(p)^* \to M^*$ is a continuous map onto the 2-disk M^*. As above, exp*: $\tilde{D}(p)^* \cap \tilde{U}^* \to U^*$ and exp*: $\tilde{D}(p)^* \cap \tilde{B}^* \to B^*$. Also, $\operatorname{int}(\tilde{D}(p)^*)$ must be contained in $\tilde{U}^*$ since it is mapped into $\operatorname{int} M^* = U^*$ by exp*. We conclude that $\operatorname{int} \tilde{C}(p)^* \subset \tilde{U}^*$ also, because $\tilde{D}(p)^*$ is a cone on the interval $\tilde{C}(p)^*$. The same reasoning shows that $\partial\tilde{D}(p)^* - \operatorname{int} \tilde{C}(p)^* \subset \tilde{B}^*$. Note that exp* projects the interval $\partial\tilde{D}(p)^* - \operatorname{int} \tilde{C}(p)^*$ onto the circle B^* by identifying the two endpoints of the interval.

We need to establish that exp* is locally one-to-one on the arc $\tilde{C}(p)^*$. Suppose this is not the case. Then there are points x, y_i, z_i in $\tilde{C}(p)$ with $Gy_i \neq Gz_i$, $\exp y_i = \exp z_i$, and elements $g_i, h_i \in G$, so that $g_i y_i \to x$ and $h_i z_i \to x$ as $i \to \infty$. Since G is compact, by choosing subsequences it suffices to assume that $g_i \to g$ and $h_i \to h$ as $i \to \infty$. If $g = h$ then $y_i \to z_i$ and exp is not one-to-one in a neighbourhood of x. This contradicts the hypothesis that there are no conjugate points in $\tilde{C}(p)$. Hence, $g \neq h$. Also, $y_i \to g^{-1}x$ and $z_i \to h^{-1}x$ as $i \to \infty$, so $\exp g^{-1}x = \exp h^{-1}x$, i.e., $\exp x = \exp gh^{-1}x = gh^{-1} \exp x$. This proves that $\exp x$ has a nontrivial isotropy subgroup and, hence, belongs to an exceptional orbit, contradicting $E^* = \varnothing$. Therefore, exp* restricted to $\tilde{C}(p)^*$ is locally one-to-one.

To complete the proof of the Theorem, we apply a similar argument to Theorem 5.1 of **[10]** (cf. **[5]** also) to conclude that there is a contradiction, since $M^* - C(p)^*$ is connected but exp*: $\tilde{C}(p)^* \to M^*$ is locally one-to-one with image $C(p)^*$. ($C(p)^*$ must be a tree, and, hence, exp* cannot be locally one-to-one at the preimage of a vertex of this tree in $\operatorname{int} M^*$.)

4. For completeness we note the following simple result when there is a codimension-one isometry group fixing a point.

PROPOSITION. *Suppose M^n is a compact, connected, C^∞ Riemannian manifold, there is a compact Lie group G of isometries of M which fix $p \in M$, and the principal orbits have codimension one. Then either the conjugate and cut loci of p intersect, or M is diffeomorphic to $\mathbf{R}P^n$.*

PROOF. Clearly G acts transitively on $\tilde{C}(p)$, so $\tilde{C}(p) = \{x \in M_p \colon \|x\| = k\}$ for some constant k. By Lemma 5.6 of **[4]** either every point of $C(p)$ is a conjugate point of p, or $\exp x = \exp y$ for $x, y \in C(p)$ if and only if $x = -y$. In the latter case exp: $\tilde{D}(p) \to M$ gives a diffeomorphism $\phi\colon \mathbf{R}P^n \to M$ by identification of $\mathbf{R}P^n$ with $\tilde{D}(p)/\sim$, where $x \sim y$ if and only if $x = -y$ and $\|x\| = k$.

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