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MINIMAL DISKS AND COMPACT HYPERSURFACES IN EUCLIDEAN SPACE

JOHN DOUGLAS MOORE AND THOMAS SCHULTE

ABSTRACT. Let M^n be a smooth connected compact hypersurface in $(n + 1)$-dimensional Euclidean space E^{n+1}, let A^{n+1} be the unbounded component of $E^{n+1} - M^n$, and let $\kappa_1 \leqslant \kappa_2 \leqslant \cdots \leqslant \kappa_n$ be the principal curvatures of M^n with respect to the unit normal pointing into A^{n+1}. It is proven that if $\kappa_2 + \cdots + \kappa_n < 0$, then A^{n+1} is simply connected.

1. Introduction. Recently the theory of minimal surfaces has yielded many striking results relating topology to curvature of Riemannian manifolds. We are interested in applying minimal surfaces to extrinsic problems which relate topology to curvature of submanifolds of low codimension in Euclidean space.

The simplest case is that of a smooth connected compact hypersurface M^n lying in $(n + 1)$-dimensional Euclidean space E^{n+1}. Such a hypersurface divides E^{n+1} into an unbounded connected open region A^{n+1} and a bounded region B^{n+1}. In this case, the basic local invariants are the principal curvatures $\kappa_1 \leqslant \kappa_2 \leqslant \cdots \leqslant \kappa_n$ with respect to the unit normal pointing into A^{n+1}. Although the principal curvatures are extrinsic, they are completely determined up to sign by the intrinsic Riemann-Christoffel curvature tensor of M^n, when the rank of the curvature is at least three. In this note we will apply techniques of Courant and Davids **[CD]** and Meeks and Yau **[MY**, Theorem 1] to solve a free boundary value problem for minimal disks, and as a consequence will prove the following

THEOREM. *Let M^n be a smooth connected compact hypersurface in E^{n+1}, and let A^{n+1} be the unbounded component of $E^{n+1} - M^n$, where $n \geqslant 2$. If $\kappa_2 + \cdots + \kappa_n < 0$, then A^{n+1} is simply connected.*

Note that if $n = 2$, the hypotheses imply that M^n is convex, from which one easily concludes that A^{n+1} is simply connected. Therefore, throughout the remainder of the article, we will assume that $n \geqslant 3$.

It should be mentioned that by an argument quite different from the one given here, Howard and Wei **[HW**, Theorem 5] show that if, in addition to our hypotheses, the principal curvatures of M^n satisfy the condition $\kappa_n + (\kappa_2 + \cdots + \kappa_n) < \kappa_1$, then $\pi_1(M) = 0$. Moreover, the referee informs us that very recently Ji-ping Sha has

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obtained results which imply our theorem by different methods, which are to appear in his thesis at Stony Brook.

2. Second variation. The proof of the Theorem will utilize the formula for second variation of area for a minimal disk in E^N whose boundary is constrained to lie in a given compact submanifold M^n of E^N.

For each $t \in (-\varepsilon, \varepsilon)$ let $X_t \colon \overline{D} \to E^N$ be a smooth map, depending smoothly on t, where $\overline{D}$ is the closed unit disk in the complex plane, and suppose that each X_t maps the boundary ∂D of the unit disk into M^n. Then the variation field

$$V = \frac{\partial}{\partial t}(X_t)|_{t=0}$$

is tangent to M along $X_0(\partial D)$. (Conversely, given any smooth $V \colon \overline{D} \to E^N$ which is tangent to M along ∂D, we can construct a corresponding family $\{X_t \colon t \in (-\varepsilon, \varepsilon)\}$ of smooth maps of $\overline{D}$ into E^N such that $X_t(\partial D) \subseteq M$.) Suppose that $X_0 \colon \overline{D} \to E^N$ is a branched conformal minimal immersion and that the variation field V is perpendicular to $X_0(\overline{D})$. If

$$A(X_t) = \text{area of } X_t(\overline{D}),$$

we claim that

$$\frac{d^2}{dt^2}(A(X_t))|_{t=0} = I(V,V),$$

where

$$I(V,V) = \int_D \left\{ \left\|(dV)^{\perp}\right\|^2 - \left\|(dV)^{\mathrm{T}}\right\|^2 \right\} dA + \int_{\partial D} \alpha(V,V) \cdot \nu \, ds. \tag{1}$$

In this formula, dV is the usual differential of the E^N-valued function V,

$$(dV)^{\mathrm{T}} = \text{component of } dV \text{ tangential to } X_t(D),$$
$$(dV)^{\perp} = \text{component of } dV \text{ perpendicular to } X_t(D),$$

α is the second fundamental form of M in E^N (see [**KN**, vol. II, p. 10]), and ν is the unit normal to M in E^N which points away from $X_0(\overline{D})$. (Since the Gauss map of a branched conformal minimal immersion extends to a smooth map at the branch points, $(dV)^{\mathrm{T}}$, $(dV)^{\perp}$ and ν are well defined at the branch points of X_0.)

If X_0 has no branch points and the variation field V vanishes on ∂D, (1) is a special case of the second variation formula found in Lawson [**L**, p. 49].

For completeness, we sketch a derivation of (1) in two steps, the first step being a derivation of the slightly easier formula for second variation of energy. If $(u, v) = (u^1, u^2)$ are standard coordinates on D, the energy of X_t is

$$E(X_t) = \frac{1}{2}\int_D \left(\frac{\partial X_t}{\partial u} \cdot \frac{\partial X_t}{\partial u} + \frac{\partial X_t}{\partial v} \cdot \frac{\partial X_t}{\partial v} \right) du\, dv.$$

Differentiation under the integral sign and integration by parts yields

$$\frac{d^2}{dt^2}(E(X_t)) = \int_D \sum_i \left|\frac{\partial^2 X_t}{\partial t \partial u^i}\right|^2 du\,dv + \int_D \sum_i \left[\frac{\partial^3 X_t}{\partial t^2 \partial u^i} \cdot \frac{\partial X_t}{\partial u^i}\right] du\,dv$$

$$= \int_D \sum_i \left|\frac{\partial}{\partial u^i}\left(\frac{\partial X_t}{\partial t}\right)\right|^2 du\,dv - \int_D \frac{\partial^2 X_t}{\partial t^2} \cdot \left[\frac{\partial^2 X_t}{\partial u^2} + \frac{\partial^2 X_t}{\partial v^2}\right] du\,dv$$

$$+ \int_{\partial D} \frac{\partial^2 X_t}{\partial t^2} \cdot \left(\frac{\partial X_t}{\partial u} dv - \frac{\partial X_t}{\partial v} du\right).$$

Under the assumption that X_0 is harmonic, the second term drops out at $t = 0$, leaving

$$\text{(2)} \qquad \frac{d^2}{dt^2}(E(X_t))|_{t=0} = \int_D \|dV\|^2 dA + \int_{\partial D} \alpha(V, V) \cdot \nu\, ds.$$

The formula for second variation of area is a direct consequence of (2) and the formula

$$\text{(3)} \qquad \frac{d^2}{dt^2}(E(X_t) - A(X_t))|_{t=0} = \int_D 2\left\|(dV)^{\mathrm{T}}\right\|^2 dA,$$

which we claim holds under the assumption that V is perpendicular to $X_0(D)$. Indeed if we set $g_{ij} = (\partial X_t/\partial u^i) \cdot (\partial X_t/\partial u^j)$, (g^{ij}) the matrix inverse to (g_{ij}) and $g = \det(g_{ij})$, then a standard calculation shows that

$$\text{(4)} \qquad \frac{\partial}{\partial t}(\sqrt{g}) = \frac{1}{2}\sqrt{g} \sum_{i,j=1}^{2} g^{ij}\left(\frac{\partial g_{ij}}{\partial t}\right).$$

Note that, at $t = 0$, $g_{ij} = \lambda \delta_{ij}$, and

$$\frac{\partial}{\partial t}(\sqrt{g})|_{t=0} = \frac{1}{2}\left(\frac{\partial}{\partial t}\right)(g_{11} + g_{22})|_{t=0} = \sum_i \frac{\partial V}{\partial u^i} \cdot \frac{\partial X_0}{\partial u^i}.$$

Since V is perpendicular to $X_0(D)$ and X_0 is harmonic,

$$\text{(5)} \qquad \frac{\partial}{\partial t}(\sqrt{g})|_{t=0} = -V \cdot \left(\frac{\partial^2 X_0}{\partial u^2} + \frac{\partial^2 X_0}{\partial v^2}\right) = 0.$$

Differentiation of (4) yields

$$\frac{\partial^2}{\partial t^2}(\sqrt{g}) = \frac{1}{2}\left(\frac{\partial \sqrt{g}}{\partial t}\right) \sum g^{ij}\left(\frac{\partial g_{ij}}{\partial t}\right) + \frac{1}{2}\sqrt{g} \sum \left(\frac{\partial g^{ij}}{\partial t}\right)\left(\frac{\partial g_{ij}}{\partial t}\right)$$

$$+ \frac{1}{2}\sqrt{g} \sum g^{ij}\left(\frac{\partial^2 g_{ij}}{\partial t^2}\right).$$

Evaluation at $t = 0$ and application of (5) gives

$$\frac{\partial^2}{\partial t^2}(\sqrt{g})|_{t=0} = \frac{1}{2}\left(\frac{\partial^2}{\partial t^2}\right)(g_{11} + g_{22})|_{t=0} - \frac{1}{2}\sqrt{g} \sum \left(\frac{\partial g_{ij}}{\partial t}\right)\left(\frac{\partial g_{kl}}{\partial t}\right) g^{ik} g^{jl}|_{t=0}.$$

But

$$\frac{\partial g_{ij}}{\partial t}\Big|_{t=0} = \frac{\partial V}{\partial u^i}\cdot\frac{\partial X_0}{\partial u^j} + \frac{\partial V}{\partial u^j}\cdot\frac{\partial X_0}{\partial u^i}$$

$$= -2V\cdot\frac{\partial^2 X_0}{\partial u^i\partial u^j} = 2\sum g_{ik}\,\big((k,j)\text{-component of }(dV)^{\mathrm{T}}\big),$$

so

$$\frac{\partial^2}{\partial t^2}(\sqrt{g})|_{t=0} = \frac{1}{2}\left(\frac{\partial^2}{\partial t^2}\right)(g_{11}+g_{22})|_{t=0} - 2\|(dV)^{\mathrm{T}}\|^2\sqrt{g}\,|_{t=0},$$

from which the desired formula (3) follows by integration.

3. Existence of a minimal disk. We now assume the setup of the theorem: M^n is a smooth connected compact hypersurface in E^{n+1}, whose complement is the disjoint union of an unbounded connected region A^{n+1} and a bounded region B^{n+1}. Throughout this section we assume that $\pi_1(\bar{A}, x_0) \neq 0$, where $\bar{A}$ = closure of A and $x_0 \in M$. It follows from the homotopy exact sequence

$$\cdots \to \pi_2(E^{n+1}, x_0) \to \pi_2(E^{n+1}, \bar{A}, x_0) \to \pi_1(\bar{A}, x_0) \to \pi_1(E^{n+1}, x_0) \to \cdots$$

that $\pi_2(E^{n+1}, \bar{A}, x_0) \neq 0$. Let Ω_0 be the space of smooth maps $X\colon \bar{D} \to E^{n+1}$ such that $X(\partial D) \subseteq \bar{A}$, and the mapping of pairs $X\colon (\bar{D}, \partial D) \to (E^{n+1}, \bar{A})$ is not homotopic to a map taking $\bar{D}$ into $\bar{A}$. (In this case, we say that X represents a nontrivial element in the set $\pi_2(E^{n+1}, \bar{A})$ of free homotopy classes.)

Let $\mu_0 = \inf\{E(X)\colon X \in \Omega_0\}$, the infimum of energy. We claim that it follows from the arguments in Courant and Davids **[CD]**, together with boundary regularity results of Jäger **[J]**, that there exists an element $X_0 \in \Omega_0$ such that $E(X_0) = \mu_0$.

To prove the existence of $X_0 \in \Omega_0$ realizing the minimum energy, it will be convenient to utilize additional function spaces Ω_1, Ω_2 and Ω_3. Let Ω_1 be the space of smooth maps $X\colon \bar{D} \to E^{n+1}$ such that $X(\partial D) \subseteq M$ and X represents a nontrivial element in $\pi_2(E^{n+1}, \bar{A})$, and let $\mu_1 = \inf\{E(X)\colon X \in \Omega_1\}$. Clearly $\mu_1 \geqslant \mu_0$, and we claim that, in fact, $\mu_1 = \mu_0$. Indeed, if $X \in \Omega_0$ and $E(X) < \mu_0 + \varepsilon$, we can put X in general position with respect to M by means of an arbitrarily small perturbation keeping the energy $< \mu_0 + \varepsilon$ and $X \in \Omega_0$, so that X is an immersion, $X(\partial D)$ does not intersect M, and $X(\text{int } D)$ intersects M in a finite number of circles. D is then divided into the disjoint union of three sets $\tilde{A} = X^{-1}(A)$, $\tilde{B} = X^{-1}(B)$ and $\tilde{M} = X^{-1}(M)$ (see Figure 1).

$\tilde{M}$ consists of a finite collection of circles $C_1, \ldots, C_l$, each C_i bounding a disk D_i contained in D. We partially order these circles by setting $C_i < C_j \Leftrightarrow C_i \subseteq D_j$. If $C_1, \ldots, C_k$ are the circles which are maximal with respect to this partial ordering, and

$$X_i\colon \bar{D} \to E^{n+1} \text{ is a reparametrization of } X|\bar{D}_i \quad \text{for } 1 \leqslant i \leqslant k,$$

then it is easily seen that at least one of these X_i's represents a nontrivial element in $\pi_2(E^{n+1}, \bar{A})$ (which is isomorphic to the set $\pi_1(\bar{A})$ of free homotopy classes of loops in $\bar{A}$). Thus $X_i \in \Omega_1$ and $E(X_i) < \mu_0 + \varepsilon$. Since ε was arbitrary, we conclude that $\mu_1 = \mu_0$.

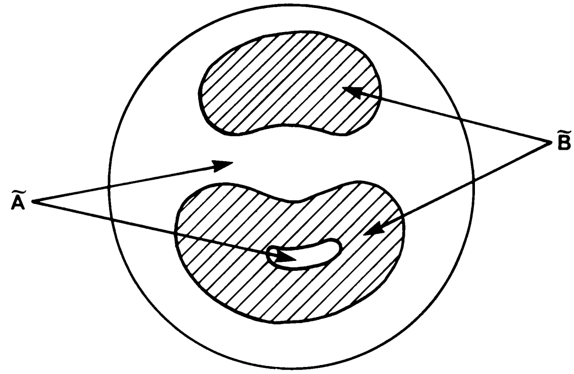

FIGURE 1

For $\varepsilon > 0$, let $M(\varepsilon) = \{p \in E^{n+1}: d(p, M) < \varepsilon\}$ and note that M is a deformation retract of $M(\varepsilon)$ for ε sufficiently small, say for $\varepsilon \leqslant \varepsilon_0$, where ε_0 is a fixed positive number. Let Ω_2 be the collection of smooth maps from the *open* unit disk D into E^{n+1} such that:

(i) $E(X)$ is finite.

(ii) given ε, $0 < \varepsilon < \varepsilon_0$, there exists an r, $0 < r < 1$, such that $X(D - D_r) \subseteq M(\varepsilon)$, where $D_r = \{(u, v) \in D: u^2 + v^2 < r^2\}$, and

(iii) $X|D_r$ represents a nontrivial element of $\pi_2(E^{n+1}, A \cup M(\varepsilon))$.

Let $\mu_2 = \inf\{E(X): X \in \Omega_2\}$.

Finally let Ω_3 be the collection of sequences $\{X_i: i = 1, 2, 3, \dots\}$ of smooth maps $X_i: D \to E^{n+1}$ such that:

(i) $E(X_i)$ is finite for each i,

(ii) given ε, $0 < \varepsilon < \varepsilon_0$, there exists a positive integer $N(\varepsilon)$ such that, for each $i > N(\varepsilon)$, $X_i(D - D_r) \subseteq M(\varepsilon)$ for r sufficiently near 1 (where r depends on i), and

(iii) $X_i|D_r$ represents a nontrivial element of $\pi_2(E^{n+1}, A \cup M(\varepsilon))$.

If $\{X_i\} \in \Omega_3$, let $E(\{X_i\}) = \liminf\{E(X_i): i = 1, 2, 3, \dots\}$ and set

$$\mu_3 = \inf\{E(\{X_i\}): \{X_i\} \in \Omega_3\}.$$

Clearly $\mu_3 \leqslant \mu_2 \leqslant \mu_1$, and we will show that, in fact, $\mu_1 = \mu_2 = \mu_3$.

One easily constructs an element $\{X_i\} \in \Omega_3$ such that $E(X_i) \to \mu_3$. By an arbitrarily small alteration of X_i, we can arrange that $X_i|\partial D_{r(i)}$ is an imbedding, where $r(i)$ is chosen to approach 1 as $i \to \infty$ with $X_i(\partial D_{r(i)})$ a noncontractible curve in $A \cup M(\varepsilon)$. Let $Y_i: D \to E^{n+1}$ be the Douglas-Courant solution to the Plateau problem with boundary curve $X(\partial D_{r(i)})$. Since $Y_i(\partial D)$ is not contractible in $A \cup M(\varepsilon)$, Y_i takes some point of D into $B - (B \cap M(\varepsilon))$, and after composing with a linear fractional transformation, we can arrange that $Y_i(0) \in B - (B \cap M(\varepsilon))$. Note that $\{Y_i\} \in \Omega_3$ and $E(Y_i) \to \mu_3$.

The Y_i's are bounded harmonic maps; hence a subsequence of them, still denoted by $\{Y_i\}$, converges uniformly on compact subsets of the open disk D to a harmonic map $X_0: D \to E^{n+1}$. By Theorem 1 of [C], $X_0(D - D_r) \subseteq M(\varepsilon)$ for r sufficiently close to 1. Since $Y_i(0) \in B - (B \cap M(\varepsilon))$, X_0 is not constant. By lower semicontinuity of the Dirichlet integral, $E(X_0) \leqslant \mu_3$ and hence $\mu_3 > 0$. Moreover,

$$(6) \qquad \frac{1}{2}\int_{D_{1/2}}\left\{\left|\frac{\partial Y_i}{\partial u}\right|^2 + \left|\frac{\partial Y_i}{\partial v}\right|^2\right\} du\, dv \geqslant c \quad \text{for some } c > 0,$$

for all sufficiently large i.

We next show that if r is sufficiently near 1, $X_0 | D_r$ represents a nontrivial element of $\pi_2(E^{n+1}, A \cup M(\varepsilon))$, and hence $X_0 \in \Omega_2$. Let (r, θ) be the usual polar coordinates in the disk and observe that, for i large

$$\frac{1}{2}\int_{D}\left\{\left|\frac{\partial Y_i}{\partial u}\right|^2 + \left|\frac{\partial Y_i}{\partial v}\right|^2\right\} du\, dv \leqslant 2\mu_3$$

$$\Rightarrow \int_0^{2\pi}\int_{1/2}^{1}\left|\frac{\partial Y_i}{\partial r}\right|^2 dr\, d\theta \leqslant 2\int_{D - D_{1/2}}\left|\frac{\partial Y_i}{\partial r}\right|^2 r\, dr\, d\theta \leqslant 8\mu_3,$$

so that for some θ_i

$$\int_\rho^1 \left|\frac{\partial Y_i}{\partial r}(r, \theta_i)\right|^2 dr \leqslant \frac{4\mu_3}{\pi} \quad \text{for } \frac{1}{2} < \rho < 1.$$

It follows from this and the Cauchy-Schwarz inequality that

$$(7) \qquad |Y_i(1, \theta_i) - Y_i(\rho, \theta_i)| \leqslant k\sqrt{1-\rho} \quad \text{for some } k > 0.$$

If $X_0 \notin \Omega_2$, we can construct sequences $\varepsilon_i \to 0$, $\rho_i \to 1$ ($\frac{1}{2} < \rho_i < 1$) such that $k\sqrt{1-\rho_i} < \varepsilon_i/3$ and $X_0 | \partial D_{\rho_i}$ is a contractible curve in $A \cup M(\varepsilon_i/3)$. After possibly passing to a subsequence, we can arrange that $Y_i | \partial D_{\rho_i}$ be a contractible curve in $A \cup M(2\varepsilon_i/3)$, while $Y_i | \partial D$ is a noncontractible curve in $A \cup M(2\varepsilon_i/3)$. Let $R_i = \{(r, \theta) \in D \colon \rho_i < r < 1,\ \theta \neq \theta_i\}$ be an annular region slit along a line segment S_i (see Figure 2). It follows from (7) that $Y_i(S_i) \subseteq A \cup M(\varepsilon_i)$, so that if $Z_i \colon D \to E^{n+1}$ is a reparametrization of $Y_i | R_i$, $Z_i | \partial D$ is noncontractible in $A \cup M(\varepsilon_i)$, and thus $\{Z_i\} \in \Omega_3$. On the other hand, it follows from (6) that

$$\frac{1}{2}\int_{R_i}\left\{\left|\frac{\partial Y_i}{\partial u}\right|^2 + \left|\frac{\partial Y_i}{\partial v}\right|^2\right\} du\, dv \leqslant E(Y_i) - c,$$

and hence

$$\mu_3 \leqslant \liminf E(Z_i) \leqslant \liminf E(Y_i) - c \leqslant \mu_3 - c,$$

a contradiction.

Thus X_0 is indeed an element of Ω_2 and $E(X_0) = \mu_2$. X_0 is harmonic and a standard argument shows that it is conformal and minimizes area. It now follows from Jäger's boundary regularity theorem [**J**, Theorem 3] that X_0 extends to a smooth map $\overline{X}_0 \colon \overline{D} \to E^{n+1}$ such that $\overline{X}_0(\partial D) \subseteq M$. In other words, $\overline{X}_0 \in \Omega_1$ and $E(\overline{X}_0) = \mu_1$. Since $\Omega_1 \subseteq \Omega_0$, $\overline{X}_0 \in \Omega_0$ and $E(\overline{X}_0) = \mu_0$.

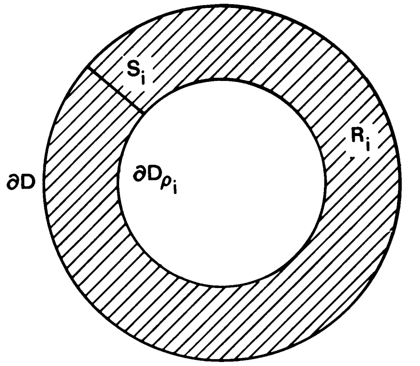

FIGURE 2

4. Conclusion of the proof. We suppose that A^{n+1} is not simply connected and derive a contradiction. Let X_0: $D \to E^{n+1}$ be the element of $\Omega_0 \cap \Omega_1$ which minimizes energy and area which we constructed in §3, a conformal branched minimal immersion. It is crucial to observe that if ν is the unit normal to M along $X_0(\partial D)$ which points out of $X_0(D)$, ν points into A. If not, the disk could be pushed slightly in the direction of $-\nu$, decreasing the area but remaining within Ω_0; this would contradict the fact that X_0 minimizes area in Ω_0.

Let $e_1, e_2, \ldots, e_{n+1}$ be a constant orthonormal frame for E^{n+1}. If $p \in \overline{D}$ is not a branch point of X_0, we set $e_i^{\mathrm{T}}(p) =$ component of e_i tangent to $X_0(D)$ at p, $e_i^{\perp}(p) =$ component of e_i perpendicular to $X_0(D)$ at p. As in §2, e_i^{T} and $e_i^{\perp}$ extend to smooth E^{n+1}-valued functions on D, because the Gauss map extends smoothly to the branch points.

Now we apply the second variation formula (1) to obtain

$$\sum_{i=1}^{n+1} I(e_i^{\perp}, e_i^{\perp}) = \int_D \sum_{i=1}^{n+1} \left(\left\| d(e_i^{\perp})^{\perp} \right\|^2 - \left\| d(e_i^{\perp})^{\mathrm{T}} \right\|^2 \right) dA + \int_{\partial D} \sum_{i=1}^{n+1} \alpha(e_i^{\perp}, e_i^{\perp}) \cdot \nu \, ds. \tag{8}$$

Since e_i is constant, $d(e_i^{\mathrm{T}}) = -d(e_i^{\perp})$, and hence

$$\sum_{i=1}^{n+1} \left\| d(e_i^{\perp})^{\perp} \right\|^2 = \sum_{i=1}^{n+1} \left\| d(e_i^{\mathrm{T}})^{\perp} \right\|^2 = \sum_{i,j=1}^{n+1} \left[d(e_i^{\mathrm{T}}) \cdot e_j^{\perp} \right]^2$$
$$= \sum_{i,j=1}^{n+1} \left[-e_i^{\mathrm{T}} \cdot d(e_j^{\perp}) \right]^2 = \sum_{j=1}^{n+1} \left\| d(e_j^{\perp})^{\mathrm{T}} \right\|^2.$$

Thus (8) simplifies to become

$$\sum_{i=1}^{n+1} I(e_i^{\perp}, e_i^{\perp}) = \int_{\partial D} \sum_{i=1}^{n+1} \alpha(e_i^{\perp}, e_i^{\perp}) \cdot \nu \, ds. \tag{9}$$

Given a point $p \in \partial D$, we can choose a new orthonormal frame $(\tilde{e}_1, \dots, \tilde{e}_{n+1})$ for $\mathbf{E}^{n+1}$ so that $\tilde{e}_n$ is tangent to ∂D and $\tilde{e}_{n+1}$ is perpendicular to M. The new frame is related to the old by a transformation

$$e_i = \sum_{j=1}^{n+1} b_{ij} \tilde{e}_j,$$

where (b_{ij}) is an $(n+1) \times (n+1)$ orthogonal matrix. Hence

$$\sum_{i=1}^{n+1} \alpha\left(e_i^{\perp}, e_i^{\perp}\right) = \sum_{i,j,k=1}^{n+1} b_{ij} b_{ik} \alpha\left(\tilde{e}_j^{\perp}, \tilde{e}_k^{\perp}\right) = \sum_{j=1}^{n+1} \alpha\left(\tilde{e}_j^{\perp}, \tilde{e}_j^{\perp}\right) = \sum_{i=1}^{n-1} \alpha(\tilde{e}_i, \tilde{e}_i),$$

from which it follows that

$$\sum_{i=1}^{n+1} \alpha\left(e_i^{\perp}, e_i^{\perp}\right) \cdot \nu = \sum_{i=1}^{n-1} \alpha(\tilde{e}_i, \tilde{e}_i) \cdot \nu \leqslant h - \kappa_1.$$

Now (9) yields the inequality

$$\sum_{i=1}^{n+1} I\left(e_i^{\perp}, e_i^{\perp}\right) \leqslant \int_{\partial D} (h - \kappa_1)\, ds.$$

By hypothesis, $h - \kappa_1 < 0$, so there exists at least one e_i with $I(e_i^{\perp}, e_i^{\perp}) < 0$, and a variation in this direction will decrease area. This contradicts the fact that our disk has least area, and the theorem is proven.

REFERENCES

[C] R. Courant, *The existence of a minimal surface of least area bounded by prescribed Jordan arcs and prescribed surfaces*, Proc. Nat. Acad. Sci. U.S.A. **24** (1938), 97–101.

[CD] R. Courant and N. Davids, *Minimal surfaces spanning closed manifolds*, Proc. Nat. Acad. Sci. U.S.A. **26** (1940), 194–199.

[HW] R. Howard and H. Wei, *On the existence and nonexistence of stable submanifolds and currents in positively curved manifolds and the topology of submanifolds in Euclidean spaces* (to appear).

[J] W. Jäger, *Behavior of minimal surfaces with free boundaries*, Comm. Pure Appl. Math. **23** (1970), 803–818.

[KN] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vols. I, II, Wiley, New York, 1963, 1969.

[L] H. B. Lawson, *Lectures on minimal submanifolds*, Vol. 1, Publish or Perish, Berkeley, Calif., 1980.

[MY] W. H. Meeks, III, and S. T. Yau, *Topology of three-dimensional manifolds and the embedding problems in minimal surface theory*, Ann. of Math. (2) **112** (1980), 441–484.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA BARBARA, CALIFORNIA 93106 (Current address of J. D. Moore)

Current address (Thomas Schulte): Naval Weapons Center, China Lake, California 93555