DETERMINACY WITH COMPLICATED STRATEGIES

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Abstract. For any class of functions \( f \) from \( \mathbb{R} \) into \( \mathbb{R} \), \( AD(f) \) is the assertion that in every two person game on integers one of the two players has a winning strategy in the class \( f \). It is shown, in \( ZF + DC + V = L(\mathbb{R}) \), that for any \( f \) of cardinality \( \leq 2^{\aleph_0} \) (i.e. any \( f \) which is a surjective image of \( \mathbb{R} \)) \( AD(f) \) implies \( AD \) (the Axiom of Determinacy).

1. Statement of the results. Let \( \omega = \{0, 1, 2, \ldots \} \) be the set of natural numbers and let \( \mathbb{R} = \omega^\omega \) be the set of all infinite sequences from \( \omega \), called for simplicity reals in the following. Given any set \( A \subseteq \mathbb{R} \times \mathbb{R} \) we associate with \( A \) the following infinite game also denoted by \( A \): In a run of the game players I and II choose alternatively natural numbers \( x(0), y(0), x(1), y(1), \ldots \)

\[
\begin{array}{c c c c}
I & x(0) & x(1) & \cdots \\
II & y(0) & y(1) & \\
\end{array}
\]

and player I wins this run if \( (x, y) \in A \); otherwise II wins. A strategy for player I is a map \( \sigma: \omega^\omega \to \omega \), where \( \omega^\omega \) is the set of all finite sequences from \( \omega \). If II plays \( (y(0), y(1), \ldots) \) then I follows the strategy \( \sigma \) if he responds by playing \( x(0), x(1), \ldots \), where \( x(n) = \sigma(y|n) \), with \( y|n = (y(0), \ldots, y(n-1)) \). The strategy \( \sigma \) is winning for player I in the game \( A \), if for all \( y \in \mathbb{R} \) when II plays \( y \) and I plays \( x \) following \( \sigma \) then \( (x, y) \in A \). Similarly we define the concept of a strategy \( \tau \) for player II and what it means to be winning for him. The set \( A \) is determined if either player I or II has a winning strategy for the game \( A \). The Axiom of Determinacy (AD) is the assertion that all sets \( A \subseteq \mathbb{R} \times \mathbb{R} \) are determined.

If \( \sigma \) is a strategy for player I, then \( \sigma \) can be viewed as a function \( \sigma^*: \mathbb{R} \to \mathbb{R} \) given by \( \sigma^*(y) = x \), where \( x(n) = \sigma(y|n) \). Similarly if \( \tau \) is a strategy for II we denote by \( \tau^* \) the corresponding function. Thus AD can be rewritten as follows:

For all \( A \subseteq \mathbb{R} \times \mathbb{R} \), either there is a strategy \( \sigma^* \) such that \( \forall y(\sigma^*(y), y) \in A \) or else there is a strategy \( \tau^* \) such that \( \forall x(x, \tau^*(x)) \notin A \).

The functions of the form \( \sigma^* \) or \( \tau^* \) are clearly continuous functions on \( \mathbb{R} \) of a very special kind, i.e. Lipschitz. What if we consider instead "strategies" which are more complicated? To make this question precise let us give the following definition.

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**Definition.** Let $\mathcal{Y}$ be a collection of functions from $\mathbb{R}$ into $\mathbb{R}$. By $\text{AD}(\mathcal{Y})$ we abbreviate the statement:

For all $A \subseteq \mathbb{R} \times \mathbb{R}$ either there is $f \in \mathcal{Y}$ with $\forall y (f(y), y) \in A$ or there is $g \in \mathcal{Y}$ with $\forall x (x, g(x)) \notin A$.

A geometric reformulation of $\text{AD}(\mathcal{Y})$ is the following: If $P \subseteq \mathbb{R} \times \mathbb{R}$ is a relation and $h: \mathbb{R} \to \mathbb{R}$ is a function, then we say that $h$ uniformizes $P$ in the $y$-direction if $\forall x (x, h(x)) \in P$. We say that $h$ uniformizes $P$ in the $x$-direction if $\forall y (h(y), y) \in P$. Then $\text{AD}(\mathcal{Y})$ asserts that for any $A \subseteq \mathbb{R} \times \mathbb{R}$ there is a function in $\mathcal{Y}$ which uniformizes $A$ in the $x$-direction or there is a function in $\mathcal{Y}$ which uniformizes $(\mathbb{R} \times \mathbb{R}) - A$ in the $y$-direction.

Clearly $\text{AD}$ implies $\text{AD}(\mathcal{Y})$ for any $\mathcal{Y}$ containing the Lipschitz functions. Mycielski [7] has considered first the question of the strength of $\text{AD}$(Continuous). Later H. Friedman [2] asked more generally about $\text{AD}$(Borel), $\text{AD}$(Projective), etc. A. Blass [1] proved that $\text{AD}$(Continuous) $\Rightarrow$ $\text{AD}$ and Kunen [6] strengthened it substantially by showing that $\text{AD}$(Borel) $\Rightarrow$ $\text{AD}$, and indeed that $\text{AD}(\Delta^1_2) \Rightarrow AD$. Some further results have been obtained in [3].

We prove in this paper an almost optimal generalization of these results, at least in the case $V = L(\mathbb{R})$, where $L(\mathbb{R})$ is the smallest inner model of ZF containing $\mathbb{R}$. First let us say that a class $\mathcal{Y}$ of functions from $\mathbb{R}$ into $\mathbb{R}$ has *cardinality* $\leq 2^{\aleph_0}$ if there is a surjection $p: \mathbb{R} \to \mathcal{Y}$. Then we have (assuming $\mathcal{Y} \supseteq$ Lipschitz)

**Theorem (ZF + DC).** (a) For any class of functions $\mathcal{Y}$ (from $\mathbb{R}$ into $\mathbb{R}$) which has cardinality $\leq 2^{\aleph_0}$, $\text{AD}(\mathcal{Y}) \Rightarrow L(\mathbb{R}) \models \text{AD}$. In particular, if $V = L(\mathbb{R})$ and $\mathcal{Y}$ has cardinality $\leq 2^{\aleph_0}$, $\text{AD}(\mathcal{Y}) \Rightarrow \text{AD}$.

(b) If $\mathcal{Y}$ is a class of functions, $\mathcal{Y} \subseteq L(\mathbb{R})$ and $\mathcal{Y}$ has cardinality $\leq 2^{\aleph_0}$, $\text{AD}(\mathcal{Y}) \Rightarrow \text{AD}$.

**Corollary.** The following two theories are equiconsistent:

(i) $\text{AD}$.

(ii) There is a class $\mathcal{Y}$ of functions (from $\mathbb{R}$ into $\mathbb{R}$) of cardinality $\leq 2^{\aleph_0}$ such that $\text{AD}(\mathcal{Y})$ holds.

**Remarks.** (1) Clearly if $\mathcal{Y}$ = the class of all continuous, $\Delta^1_2$, projective, etc. functions, then $\mathcal{Y} \subseteq L(\mathbb{R})$ and $\mathcal{Y}$ has cardinality $\leq 2^{\aleph_0}$, so (b) is a strengthening of the earlier results, and provides a full answer to the question of H. Friedman.

(2) If $\mathcal{Y}$ = the class of all functions on $\mathbb{R}$, then the assumption that every $A \subseteq \mathbb{R} \times \mathbb{R}$ can be uniformized implies trivially $\text{AD}(\mathcal{Y})$, thus $\text{AD}(\mathcal{Y})$ is a weak hypothesis (equiconsistent with ZF).

(3) Our method of proof of (b) provides also simple proofs of some earlier results, such as $\text{AD}$(Borel) $\Rightarrow \text{AD}$. We explain this after giving the proof of the main theorem.

In conclusion, we would like to thank H. Becker for pointing out that the conditions on $\mathcal{Y}$ under which an original version of (a) was proved amount to just that $\mathcal{Y}$ has cardinality $\leq 2^{\aleph_0}$.
2. Proofs. We first prove part (a) of the theorem. Our argument combines the ideas of [4 and 5], with which we have to assume familiarity. (The proof of (b), which contains the main new idea of this paper, is however self-contained modulo just the statement of part (a).)

Let \( \mathcal{F} \) be a class of functions of cardinality \( \leq 2^{\aleph_0} \) such that \( \text{AD}(\mathcal{F}) \) holds (we assume of course throughout \( ZF + DC \)). In order to prove that \( L(\mathbb{R}) \models \text{AD} \) we will use the following result proved in [4]:

If for any \( \lambda < \Theta \) there is a cardinal \( \kappa > \lambda \) with the strong partition property \( \kappa \rightarrow (\kappa)^{\kappa} \), then \( L(\mathbb{R}) \models \text{AD} \).

So it is enough to prove that \( \forall \lambda < \Theta \exists \kappa > (\kappa \rightarrow (\kappa)^{\kappa}) \). Now in [5] it is indeed shown that

\[
\text{AD} \Rightarrow \forall \lambda < \Theta \exists \kappa > (\kappa \rightarrow (\kappa)^{\kappa}).
\]

Our main observation is that one can still prove

\[
\text{AD}(\mathcal{F}) \Rightarrow \forall \lambda < \Theta \exists \kappa > (\kappa \rightarrow (\kappa)^{\kappa}),
\]

provided \( \mathcal{F} \) has cardinality \( \leq 2^{\aleph_0} \).

Indeed fix \( \lambda < \Theta \) and let \( \pi: \mathbb{R} \rightarrow \mathcal{F} \) be a surjection. Let \( A \subseteq \mathbb{R}^3 \) be the relation that codes \( \pi \), i.e.

\[
(x, y, z) \in A \iff \pi(x)(y) = z.
\]

Let now, using the terminology of [5], \( \Gamma \) be a Spector pointclass closed under \( ^3E \) such that \( \lambda < \kappa = 0 (\Delta) (= \text{supremum of the ranks of prewellorderings in } \Delta) \) and \( A \subseteq \Delta \). Now by [5, Theorem 1.1], if we knew that \( \text{AD} \) holds then we would have that \( \kappa \) has the strong partition property. The following change in the proof of [5, Theorem 1.1] makes the same proof work under the weaker assumption that \( \text{AD}(\mathcal{F}) \) holds:

In p. 80 of [5] modify the definition of \( \Sigma^1_1(\chi) \) formulas to the obvious definition of \( \Sigma^1_1(\chi, \psi) \) formulas, where \( \chi \) denotes now a partial function from \( \mathbb{R} \times \mathbb{R} \) into \( \omega \) and \( \psi \) a partial function from \( \mathbb{R} \times \mathbb{R} \times \mathbb{R} \) into \( \omega \). Then let \( \Sigma^1_1(\chi; A) \) be \( \Sigma^1_1(\chi; \psi_A) \), where \( \psi_A \) is the characteristic function of \( A \). Replace now in the rest of the proof of [5, Theorem 1.1], \( \Sigma^1_1(\chi) \) by \( \Sigma^1_1(\chi; A) \).

We now come to the proof of (b). So assume \( \mathcal{F} \subseteq L(\mathbb{R}) \), \( \mathcal{F} \) has cardinality \( \leq 2^{\aleph_0} \) and \( \text{AD}(\mathcal{F}) \) holds. Then by (a) we surely have that at least \( \text{AD}^{L(\mathbb{R})} \) holds. In order to prove full \( \text{AD} \), consider an arbitrary game \( A \subseteq \mathbb{R} \times \mathbb{R} \). Define then the following auxiliary game \( A^* \subseteq \mathbb{R} \times \mathbb{R} \):

\[
A^*(\sigma, \tau) \iff \sigma \text{ a strategy for I } [ \tau \text{ is a strategy for II } \Rightarrow \sigma \ast \tau \in A].
\]

where for strategies \( \sigma, \tau \) for I, II, respectively, we let \( \sigma \ast \tau \) be the outcome of the run of the game in which I follows \( \sigma \) and II follows \( \tau \). Thus \( A^* \) is almost like \( A \) except that players I and II play against each other's strategies instead of individual moves. (Strategies are of course viewed as reals.)

Now by our hypothesis there is \( f \in \mathcal{F} \) such that \( \forall \tau(f(\tau), \tau) \in A^* \) or else there is \( g \in \mathcal{F} \) with \( \forall \sigma(\sigma, g(\sigma)) \notin A^* \). The two cases are similar, so let us assume that we have the first one. Put

\[
B = \{ f(\tau) \ast \tau: \tau \text{ is strategy for II} \}.
\]
Since $\forall \tau (f(\tau), \tau) \in A^*$, if $\tau$ is a strategy for II, then $f(\tau) \ast \tau \in A$, so $B \subseteq A$. But $f \in \mathcal{G} \subseteq L(\mathbb{R})$, thus $B \subseteq L(\mathbb{R})$, and so by $\text{AD}_{L(\mathbb{R})}$, $B$ is determined. If I has a winning strategy in $B$ then I clearly has a winning strategy in $A$ and we are done; else II has a winning strategy $\tau_0$ in $B$. Let $\sigma_0 = f(\tau_0)$. Then $\sigma_0 \ast \tau_0 \notin B$, but by definition of $B$ we also have $\sigma_0 \ast \tau_0 \in B$, a contradiction, and our proof is complete.

As a final illustration of this technique let us give a simple proof of Kunen's result that $\text{AD}(\text{Borel}) \implies \text{AD}$.

Assume $\text{AD}(\text{Borel})$. Then it is easy to verify that we have hyperdegree determinacy, i.e., every set of hyperdegrees contains or is disjoint from a cone of hyperdegrees. From this it follows as usual that $\aleph_1$ is measurable and thus $\Sigma^1_1$-Determinacy holds (the arguments involved here are of course due to Martin for the case of Turing degrees, and carry over trivially to hyperdegrees). Let now $A \subseteq \mathbb{R} \times \mathbb{R}$ be any game and consider the game $A^*$ as before. Using $\text{AD}(\text{Borel})$, let us say that $f$ is Borel with $\forall \tau (f(\tau), \tau) \in A^*$. Form as before the set $B = \{ f(\tau) \ast \tau : \tau \text{ is a strategy for II} \}$. Then $B \in \Sigma^1_1$, so $B$ is determined and the rest of the argument is exactly the same.

\section*{References}

2. H. Friedman, private communication, January 1975.