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DETERMINACY WITH COMPLICATED STRATEGIES

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ABSTRACT. For any class of functions $\mathfrak{F}$ from $\mathbf{R}$ into $\mathbf{R}$, AD($\mathfrak{F}$) is the assertion that in every two person game on integers one of the two players has a winning strategy in the class $\mathfrak{F}$. It is shown, in $ZF + DC + V = L(\mathbf{R})$, that for any $\mathfrak{F}$ of cardinality $\leq 2^{\aleph_0}$ (i.e. any $\mathfrak{F}$ which is a surjective image of $\mathbf{R}$) AD($\mathfrak{F}$) implies AD (the Axiom of Determinacy).

1. Statement of the results. Let $\omega = \{0, 1, 2, \dots\}$ be the set of natural numbers and let $\mathbf{R} = \omega^\omega$ be the set of all infinite sequences from ω, called for simplicity *reals* in the following. Given any set $A \subseteq \mathbf{R} \times \mathbf{R}$ we associate with A the following infinite game also denoted by A: In a run of the game players I and II choose alternatively natural numbers $x(0), y(0), x(1), y(1), \dots$

$$\begin{array}{lcccccc} \mathrm{I} & x(0) & & x(1) & & & \\ & & & & & & \cdots \\ \mathrm{II} & & y(0) & & y(1) & & \end{array}$$

and player I wins this run if $(x, y) \in A$; otherwise II wins. A strategy for player I is a map $\sigma\colon \omega^{<\omega} \to \omega$, where $\omega^{<\omega}$ = the set of all finite sequences from ω. If II plays $(y(0), y(1), \dots)$ then I follows the strategy σ if he responds by playing $x(0), x(1), \dots$, where $x(n) = \sigma(y|n)$, with $y|n = (y(0), \dots, y(n-1))$. The strategy σ is winning for player I in the game A, if for all $y \in \mathbf{R}$ when II plays y and I plays x following σ then $(x, y) \in A$. Similarly we define the concept of a strategy τ for player II and what it means to be winning for him. The set A is *determined* if either player I or II has a winning strategy for the game A. The *Axiom of Determinacy* (AD) is the assertion that all sets $A \subseteq \mathbf{R} \times \mathbf{R}$ are determined.

If σ is a strategy for player I, then σ can be viewed as a function $\sigma^*\colon \mathbf{R} \to \mathbf{R}$ given by $\sigma^*(y) = x$, where $x(n) = \sigma(y|n)$. Similarly if τ is a strategy for II we denote by τ^* the corresponding function. Thus AD can be rewritten as follows:

For all $A \subseteq \mathbf{R} \times \mathbf{R}$, either there is a strategy σ^* such that $\forall y(\sigma^*(y), y) \in A$ or else there is a strategy τ^* such that $\forall x(x, \tau^*(x)) \notin A$.

The functions of the form σ^* or τ^* are clearly continuous functions on $\mathbf{R}$ of a very special kind, i.e. Lipschitz. What if we consider instead "strategies" which are more complicated? To make this question precise let us give the following definition.

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DEFINITION. Let $\mathfrak{F}$ be a collection of functions from $\mathbf{R}$ into $\mathbf{R}$. By AD($\mathfrak{F}$) we abbreviate the statement:

For all $A \subseteq \mathbf{R} \times \mathbf{R}$ either there is $f \in \mathfrak{F}$ with $\forall y(f(y), y) \in A$ or there is $g \in \mathfrak{F}$ with $\forall x(x, g(x)) \notin A$.

A geometric reformulation of AD($\mathfrak{F}$) is the following: If $P \subseteq \mathbf{R} \times \mathbf{R}$ is a relation and h: $\mathbf{R} \to \mathbf{R}$ is a function, then we say that h *uniformizes* P *in the y-direction* if $\forall x(x, h(x)) \in P$. We say that h *uniformizes* P *in the x-direction* if $\forall y(h(y), y) \in P$. Then AD($\mathfrak{F}$) asserts that for any $A \subseteq \mathbf{R} \times \mathbf{R}$ there is a function in $\mathfrak{F}$ which uniformizes A in the x-direction or there is a function in $\mathfrak{F}$ which uniformizes $(\mathbf{R} \times \mathbf{R}) - A$ in the y-direction.

Clearly AD implies AD($\mathfrak{F}$) for any $\mathfrak{F}$ containing the Lipschitz functions. Mycielski [**7**] has considered first the question of the strength of AD(Continuous). Later H. Friedman [**2**] asked more generally about AD(Borel), AD(Projective), etc. A. Blass [**1**] proved that AD(Continuous) $\Rightarrow$ AD and Kunen [**6**] strengthened it substantially by showing that AD(Borel) $\Rightarrow$ AD, and indeed that AD($\mathbf{\Delta}^1_2$) $\Rightarrow$ AD. Some further results have been obtained in [**3**].

We prove in this paper an almost optimal generalization of these results, at least in the case $V = L(\mathbf{R})$, where $L(\mathbf{R})$ is the smallest inner model of ZF containing $\mathbf{R}$. First let us say that a class $\mathfrak{F}$ of functions from $\mathbf{R}$ into $\mathbf{R}$ has *cardinality* $\leq 2^{\aleph_0}$ if there is a surjection p: $\mathbf{R} \twoheadrightarrow \mathfrak{F}$. Then we have (assuming $\mathfrak{F} \supseteq$ Lipschitz)

THEOREM ($ZF + DC$). (a) *For any class of functions* $\mathfrak{F}$ (*from* $\mathbf{R}$ *into* $\mathbf{R}$) *which has cardinality* $\leq 2^{\aleph_0}$, AD($\mathfrak{F}$) $\Rightarrow L(\mathbf{R}) \models$ AD. *In particular, if* $V = L(\mathbf{R})$ *and* $\mathfrak{F}$ *has cardinality* $\leq 2^{\aleph_0}$, AD($\mathfrak{F}$) $\Rightarrow$ AD.

(b) *If* $\mathfrak{F}$ *is a class of functions*, $\mathfrak{F} \subseteq L(\mathbf{R})$ *and* $\mathfrak{F}$ *has cardinality* $\leq 2^{\aleph_0}$, AD($\mathfrak{F}$) $\Rightarrow$ AD.

COROLLARY. *The following two theories are equiconsistent*:

(i) AD.

(ii) *There is a class* $\mathfrak{F}$ *of functions* (*from* $\mathbf{R}$ *into* $\mathbf{R}$) *of cardinality* $\leq 2^{\aleph_0}$ *such that* AD($\mathfrak{F}$) *holds*.

REMARKS. (1) Clearly if $\mathfrak{F}$ = the class of all continuous, $\mathbf{\Delta}^1_2$, projective, etc. functions, then $\mathfrak{F} \subseteq$ L($\mathbf{R}$) and $\mathfrak{F}$ has cardinality $\leq 2^{\aleph_0}$, so (b) is a strengthening of the earlier results, and provides a full answer to the question of H. Friedman.

(2) If $\overline{\mathfrak{F}}$ = the class of *all* functions on $\mathbf{R}$, then the assumption that every $A \subseteq \mathbf{R} \times \mathbf{R}$ can be uniformized implies trivially AD($\overline{\mathfrak{F}}$), thus AD($\overline{\mathfrak{F}}$) is a weak hypothesis (equiconsistent with ZF).

(3) Our method of proof of (b) provides also simple proofs of some earlier results, such as AD(Borel) $\Rightarrow$ AD. We explain this after giving the proof of the main theorem.

In conclusion, we would like to thank H. Becker for pointing out that the conditions on $\mathfrak{F}$ under which an original version of (a) was proved amount to just that $\mathfrak{F}$ has cardinality $\leq 2^{\aleph_0}$.

2. Proofs. We first prove part (a) of the theorem. Our argument combines the ideas of [**4** and **5**], with which we have to assume familiarity. (The proof of (b), which contains the main new idea of this paper, is however self-contained modulo just the statement of part (a).)

Let $\mathfrak{F}$ be a class of functions of cardinality $\leq 2^{\aleph_0}$ such that AD($\mathfrak{F}$) holds (we assume of course throughout $ZF + DC$). In order to prove that $L(\mathbf{R}) \models$ AD we will use the following result proved in [**4**]:

If for any $\lambda < \Theta$ there is a cardinal $\kappa > \lambda$ with the strong partition property $\kappa \to (\kappa)^\kappa$, then $L(\mathbf{R}) \models$ AD.

So it is enough to prove that $\forall\lambda < \Theta\exists\kappa > (\kappa \to (\kappa)^\kappa)$. Now in [**5**] it is indeed shown that

$$\mathrm{AD} \Rightarrow \forall\lambda < \Theta\exists\kappa > \lambda\big(\kappa \to (\kappa)^\kappa\big).$$

Our main observation is that one can still prove

$$\mathrm{AD}(\mathfrak{F}) \Rightarrow \forall\lambda < \Theta\exists\kappa > \lambda\big(\kappa \to (\kappa)^\kappa\big),$$

provided $\mathfrak{F}$ has cardinality $\leq 2^{\aleph_0}$.

Indeed fix $\lambda < \Theta$ and let π: $\mathbf{R} \twoheadrightarrow \mathfrak{F}$ be a surjection. Let $A \subseteq \mathbf{R}^3$ be the relation that codes π, i.e.

$$(x, y, z) \in A \Leftrightarrow \pi(x)(y) = z.$$

Let now, using the terminology of [**5**], Γ be a Spector pointclass closed under ${}^3E$ such that $\lambda < \kappa = 0\,(\Delta)$ (= supremum of the ranks of prewellorderings in Δ) and $A \in \Delta$. Now by [**5**, Theorem 1.1], if we knew that AD holds then we would have that κ has the strong partition property. The following change in the proof of [**5**, Theorem 1.1] makes the same proof work under the weaker assumption that AD($\mathfrak{F}$) holds:

In p. 80 of [**5**] modify the definition of $\Sigma^1_1(\chi)$ formulas to the obvious definition of $\Sigma^1_1(\chi, \psi)$ formulas, where χ denotes now a partial function from $\mathbf{R} \times \mathbf{R}$ into ω and ψ a partial function from $\mathbf{R} \times \mathbf{R} \times \mathbf{R}$ into ω. Then let $\Sigma^1_1(\chi; A)$ be $\Sigma^1_1(\chi; \psi_A)$, where ψ_A is the characteristic function of A. Replace now in the rest of the proof of [**5**, Theorem 1.1], $\Sigma^1_1(\chi)$ by $\Sigma^1_1(\chi; A)$.

We now come to the proof of (b). So assume $\mathfrak{F} \subseteq L(\mathbf{R})$, $\mathfrak{F}$ has cardinality $\leq 2^{\aleph_0}$ and AD($\mathfrak{F}$) holds. Then by (a) we surely have that at least $\mathrm{AD}^{L(\mathbf{R})}$ holds. In order to prove full AD, consider an arbitrary game $A \subseteq \mathbf{R} \times \mathbf{R}$. Define then the following auxiliary game $A^* \subseteq \mathbf{R} \times \mathbf{R}$:

$A^*(\sigma, \tau) \Leftrightarrow \sigma$ a strategy for I $\wedge$ [τ is a strategy for II $\Rightarrow \sigma * \tau \in A$],
where for strategies σ, τ for I, II, respectively, we let $\sigma * \tau$ be the outcome of the run of the game in which I follows σ and II follows τ. Thus A^* is almost like A except that players I and II play against each other's strategies instead of individual moves. (Strategies are of course viewed as reals.)

Now by our hypothesis there is $f \in \mathfrak{F}$ such that $\forall\tau(f(\tau), \tau) \in A^*$ or else there is $g \in \mathfrak{F}$ with $\forall\sigma(\sigma, g(\sigma)) \notin A^*$. The two cases are similar, so let us assume that we have the first one. Put

$$B = \{f(\tau) * \tau\colon \tau \text{ is strategy for II}\}.$$

Since $\forall\tau(f(\tau),\tau)\in A^*$, if τ is a strategy for II, then $f(\tau)*\tau\in A$, so $B\subseteq A$. But $f\in\mathfrak{F}\subseteq L(\mathbf{R})$, thus $B\in L(\mathbf{R})$, and so by $\mathrm{AD}^{L(\mathbf{R})}$, B is determined. If I has a winning strategy in B then I clearly has a winning strategy in A and we are done; else II has a winning strategy τ_0 in B. Let $\sigma_0=f(\tau_0)$. Then $\sigma_0*\tau_0\notin B$, but by definition of B we also have $\sigma_0*\tau_0\in B$, a contradiction, and our proof is complete.

As a final illustration of this technique let us give a simple proof of Kunen's result that AD(Borel) $\Rightarrow$ AD.

Assume AD(Borel). Then it is easy to verify that we have hyperdegree determinacy, i.e., every set of hyperdegrees contains or is disjoint from a cone of hyperdegrees. From this it follows as usual that $\aleph_1$ is measurable and thus $\mathbf{\Sigma}^1_1$-Determinacy holds (the arguments involved here are of course due to Martin for the case of Turing degrees, and carry over trivially to hyperdegrees). Let now $A\subseteq\mathbf{R}\times\mathbf{R}$ be any game and consider the game A^* as before. Using AD(Borel), let us say that f is Borel with $\forall\tau(f(\tau),\tau)\in A^*$. Form as before the set $B=\{f(\tau)*\tau\colon\tau \text{ is a strategy for II}\}$. Then $B\in\mathbf{\Sigma}^1_1$, so B is determined and the rest of the argument is exactly the same.

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