PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 94, Number 2, June 1985

# EQUIVARIANT $K$-THEORY AND REPRESENTATIONS OF HECKE ALGEBRAS

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ABSTRACT. We construct some representations of the Hecke algebra of an affine Weyl group using equivariant $K$-theory and state a conjecture on some $q$-analogs of the Springer representations.

**1.** This paper contains a new construction of the principal series representations of the Hecke algebra of an affine Weyl group, in terms of the equivariant $K$-theory of a flag manifold. The formulas defining the simplest operators in these representations are $q$-analogs of operators considered earlier by Demazure [**1**]. The results in this paper were found during a visit to the Tata Institute of Fundamental Research, Bombay, in December 1983; I am grateful to D. N. Verma for some stimulating discussions.

**2.** We recall (cf. Segal [**4**]) that if $X$ is a compact topological space with a continuous action of a compact topological group $M'$, then the equivariant $K$-theory $K_{M'}(X)$ is defined as the Grothendieck group of the category whose objects are the $M'$-equivariant complex vector bundles on $X$ and the morphisms are $M'$-equivariant maps with locally constant rank. Then $K_{M'}(X)$ is naturally an $R_{M'}$-module where $R_{M'} = K_{M'}$ (point) is the representation ring of $M'$, i.e. the Grothendieck group of the category of finite dimensional continuous complex representations of $M'$.

**3.** We shall need a variant of this definition, in which $M'$ is replaced by a complex Lie group $M$ underlying a (not necessarily connected) reductive complex algebraic group. We assume that $M$ acts continuously on the compact topological space $X$, and we wish to define $K_M(X)$.

According to Mostow (see [**2**, Chapter XV]), $M$ has maximal compact subgroups, any two such are conjugate and any connected component of the normalizer in $M$ of a maximal compact subgroup $M'$, meets $M'$. We shall construct, for any two maximal compact subgroups $M'$ and $M''$ of $M$, a canonical isomorphism $\phi_{M',M''}: K_{M'}(X) \xrightarrow{\sim} K_{M''}(X)$ as follows. Choose $g \in M$ such that $gM''g^{-1} = M'$. If $E$ is an $M'$-equivariant vector bundle on $X$, we define a new vector bundle $g^*E$ on $X$ as the inverse image of $E$ under the map $X \to X$, $x \mapsto gx$. It is clear that $g^*E$ is naturally an $M''$-equivariant vector bundle on $X$ and $\phi_{M',M''}$ is defined by $E \mapsto g^*E$. To show that $\phi_{M',M''}$ is independent of the choice of $g$, we may assume that $M' = M''$ and that $g \in M$ normalizes $M'$; we must show that $g^*E \approx E$ as $M'$-equivariant vector bundles. Since the isomorphism class of $g^*E$ does not change when $g$ runs through a fixed connected component of the normalizer of $M'$, and since $M'$ meets each such component, we can further assume that $g \in M'$. In this

Received by the editors August 12, 1984.
1980 *Mathematics Subject Classification*. Primary 22E50; Secondary 18F25, 16A99.
[1]Supported in part by the National Science Foundation.

case, the $M'$-equivariant structure of $E$ defines an isomorphism $E_x \xrightarrow{g} E_{gx} = (g^*E)_x$ for all $x \in X$ and hence an isomorphism $g^*E \approx E$ as desired.

The isomorphisms $\phi_{M',M''}$ have an obvious transitivity property. We may therefore define $K_M(X)$ to be $\varprojlim K_{M'}(X)$ (limit over all maximal compact subgroups $M'$ of $M$, with respect to the isomorphisms $\phi_{M',M''}$). Then we have natural isomorphisms $K(X) \xrightarrow{\sim} K_{M'}(X)$ for any maximal compact subgroup $M' \subset M$. It also follows that $K_M(X)$ is naturally an $R_M$-module where $R_M = K_M(\text{point})$ is the representation ring of $M$, i.e. the Grothedieck group of the category of finite dimensional complex algebraic representations of $M$. (Note that $R_M \xrightarrow{\sim} R_{M'}$ for any maximal compact subgroup $M' \subset M$.)

**4.** We now consider a simple, simply connected complex algebraic group $G$ and $X = G/B$, where $B$ is a Borel subgroup of $G$. Then $M = G \times \mathbf{C}^*$ acts on $X$ as follows: $G$ acts by left translation and $\mathbf{C}^*$ acts trivially. We have $K_M(X) = K_G(X) \otimes R_{\mathbf{C}^*} = K_G(X) \otimes Z[q, q^{-1}]$, where $q$ is the generator of $R_{\mathbf{C}^*}$ corresponding to the identity representation $\mathbf{C}^* \xrightarrow{\sim} \mathbf{C}^*$.

Let $T$ be a maximal torus in $B$, $W$ the Weyl group of $G$ with respect to $T$, $S$ the set of simple reflections in $W$ (with respect to $B$), $P$ the lattice of weights $T \to \mathbf{C}^*$, $R \subset P$ the set of roots and $R^+$ the set of positive roots (with respect to $B$).

For each $s \in S$, let $P_s$ be the parabolic subgroup $B \cup BsB$ and let $\pi_s: X \to G/P_s$ be the natural map. There is a unique endomorphism

$$T_s: K_M(X) \to K_M(X) \tag{4.1}$$

with the following property: if $E$ is an $M$-equivariant algebraic vector bundle on $X$, then

$$E + T_sE = \pi_s^*(\pi_s)_*(E) - \pi_s^*(\pi_s)_*(E \otimes \Omega_s^1), \tag{4.2}$$

where $\Omega_s^1$ is the line bundle on $X$ of holomorphic differential 1-forms along the fibres of $\pi_s$, regarded as an $M$-equivariant bundle with the obvious action of $G$ and with the action of $\mathbf{C}^*$ given by scalar multiplication on each fibre of $\Omega_s^1$. Here $(\pi_s)_*(E)$ is the alternating sum of the higher direct images of $E$ under $\pi_s$ in the category of coherent sheaves; these higher direct images are again $M$-equivariant algebraic vector bundles on $G/P_s$, hence their alternating sum defines an element in $K_M(G/P_s)$.

For any weight $p \in P$, we define an endomorphism

$$\theta_p: K_M(X) \to K_M(X) \tag{4.3}$$

by

$$\theta_p E = E \otimes L_p^*, \tag{4.4}$$

where $L_p$ is the line bundle on $X = G/B$ associated to the homomorphism $B \to \mathbf{C}^*$ obtained by composing the projection $B \to T$ with $p: T \to \mathbf{C}^*$, and $L_p^*$ is the dual line bundle; we regard $L_p^*$ as an $M$-equivariant bundle with the obvious action of $G$ and with trivial action of $\mathbf{C}^*$.

**5.** The group structure on the lattice of weights $P$ will be written multiplicatively. The Weyl group $W$ acts naturally on $P$ ($w: p \to w(p)$) and we form the semidirect product $\tilde{W} = W \cdot P$ with $P$ normal and $w \cdot p = w(p) \cdot w$ ($w \in W$, $p \in P$).

Then $\tilde{W}$ contains the affine Weyl group as a subgroup of finite index. According to Bernstein (see [**3**, 4.4]) one can describe the Hecke algebra $\tilde{H}$ corresponding to $\tilde{W}$ as follows. It is an algebra over $\mathbf{Z}[q, q^{-1}]$ with generators $T_s$ ($s \in S$) and $\theta_p$ ($p \in P$) subject to the following relations:

$$(T_s + 1)(T_s - q) = 0 \qquad (s \in S), \tag{5.1}$$

$$T_s T_t T_s \cdots = T_t T_s T_t \cdots \tag{5.2}$$

($s \neq t \in S$; both sides have $m_{s,t}$ factors where $m_{s,t}$ = order of $st$ in $W$),

$$\theta_p \theta_{p'} = \theta_{pp'} \qquad (p, p' \in P), \tag{5.3}$$

$$T_s \theta_p = \theta_p T_s \qquad (s \in S, p \in P, sp = ps), \tag{5.4}$$

$$T_s \theta_{s(p)} T_s = q\theta_p \qquad (s \in S, p \in P, sps^{-1}p^{-1} = \alpha_s^{-1}). \tag{5.5}$$

Here $\alpha_s \in R^+ \subset P$ is the simple root corresponding to $s$. We can now state our main result.

THEOREM. *The endomorphisms $T_s$, $\theta_p$ of $K_M(X)$ defined in §4 give rise to a left $\tilde{H}$-module structure on $K_M(X)$. (The action of $\mathbf{Z}[q, q^{-1}] \subset \tilde{H}$ is defined to be the same as the restriction to $R_{\mathbf{C}^*}$ of the earlier action of $R_M$.) This $\tilde{H}$-module structure commutes with the $R_M$-module structure on $K_M(X)$.*

The proof will be given in §8.1.

**6.** We now fix a unipotent element $u \in G$. Let $\phi$: $\mathrm{SL}_2(\mathbf{C}) \to G$ be an algebraic homomorphism such that $u = \phi\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ and let $D$ be the group of diagonal matrices in $\mathrm{SL}_2(\mathbf{C})$. We define

$$M_u = \{(g, \lambda) \in G \times \mathbf{C}^* | g^{-1}ug = u^\lambda,\ g \in Z_G(\phi(D))\}.$$

(Note that any complex power of a unipotent element is well defined.) Then $M_u$ is a reductive algebraic subgroup of $G \times \mathbf{C}^*$; if $u = e$ then $M_u = M$ of §4. Let $X_u = \{g_1 B \in X | g_1^{-1} u g_1 \in B\}$; it is a closed subvariety of $X = G/B$ with an action of $M_u$ given by $(g, \lambda)\colon g_1 B \to g g_1 B$. We now fix a connected component $c$ of $M_u$ and let $M_{u,c}$ be the inverse image under $M_u \to M_u/M_u^0$ of the cyclic group in $M_u/M_u^0$ generated by the image of $c$. The $R_{M_{u,c}}$-module $K_{M_{u,c}}(X_u)$ is defined as in §2. Restriction of vector bundles gives rise to a homomorphism of $R_{M_{u,c}}$-modules

$$R_{M_{u,c}} \otimes_{R_M} K_M(X) \to K_{M_{u,c}}(X_u). \tag{6.1}$$

(We regard $R_{M_{u,c}}$ as an $R_M$-module, via the homomorphism $R_M \to R_{M_{u,c}}$ induced by the inclusion $M_{u,c} \subset M$.) Now let $\overline{R}_{M_{u,c}}$ be the ring $R_{M_{u,c}}/J$, where $J$ is the ideal of $R_{M_{u,c}}$ consisting of all $E \in R_{M_{u,c}}$ whose character is identically zero on $c$. Then from (6.1) we get a homomorphism of $\overline{R}_{M_{u,c}}$-modules

$$\overline{R}_{M_{u,c}} \otimes_{R_M} K_M(X) \to \overline{R}_{M_{u,c}} \otimes_{R_{M_{u,c}}} K_{M_{u,c}}(X_u). \tag{6.2}$$

We have a natural surjective homomorphism $M_{u,c} \to \mathbf{C}^*$, $(g, \lambda) \to \lambda$; it induces a ring homomorphism $R_{\mathbf{C}^*} \to R_{M_{u,c}}$ and we shall denote the image of $q \in R_{\mathbf{C}^*}$ in $\overline{R}_{M_{u,c}}$ again by $q$. We can now state the following conjecture.

CONJECTURE. *There exists a natural left $\tilde{H}$-module structure on*

$$\overline{R}_{M_{u,c}} \otimes_{R_{M_{u,c}}} K_{M_{u,c}}(X_u)$$

*with the following properties:*

(a) *It is compatible under* (6.2) *with the $\tilde{H}$-module structure on*

$$\overline{R}_{M_{u,c}} \otimes_{R_M} K_M(X)$$

*deduced by extension of scalars from the $\tilde{H}$-module structure on $K_M(X)$ described in the Theorem.*

(b) *The action of $\theta_p$ $(p \in P)$ on*

$$\overline{R}_{M_{u,c}} \otimes_{R_{M_{u,c}}} K_{M_{u,c}}(X_u)$$

*is by tensor product with the restriction of $L_p^*$ to $X_u$ (see* (4.4)*).*

(c) *It commutes with the $\overline{R}_{M_{u,c}}$-module structure, and $q \in \tilde{H}$ acts in the same way as $q \in \overline{R}_{M_{u,c}}$.*

**7.** Assuming the conjecture, let us consider a semisimple element $s \in G$ such that $(s, q_0) \in M_u$ for some $q_0 \in \mathbf{C}^*$. Let $c$ be the component of $M_u$ containing $s$. Let

$$\tilde{H}(q_0) = \mathbf{C} \otimes_{\mathbf{Z}[q,q^{-1}]} \tilde{H},$$

where $\mathbf{C}$ is regarded as a $\mathbf{Z}[q, q^{-1}]$-module with $q$ acting as multiplication by $q_0$. Let $h_s \colon R_{M_{u,c}} \to \mathbf{C}$ be the ring homomorphism defined by $E \to \mathrm{Tr}(s, E)$ $(E \in R_{M_{u,c}})$. This homomorphism factors through $\overline{R}_{M_{u,c}}$, since $s \in c$. Consider the tensor product

$$F_{u,s} = \mathbf{C} \otimes_{R_{M_{u,c}}} K_{M_{u,c}}(X_u),$$

where $\mathbf{C}$ is regarded as a $R_{M_{u,c}}$-module via $h_s$. Note that $F_{u,s}$ is a finite dimensional $\mathbf{C}$-vector space. (Indeed by a theorem of Segal [**4**], $K_{M_{u,c}}(X_u)$ is a finitely generated $R_{M_{u,c}}$-module.) The conjecture implies that there is a natural left $\tilde{H}(q_0)$-module structure on $F_{u,s}$. It is a $q$-analog of the $W$-representations of Springer, which were extended to $\tilde{W}$ by S. Kato (Nederl. Akad. Wetensch. Proc. Ser. A **86** (1983), 193–201).

**8. Proof of the theorem.** It is well known that the elements $L_p \in K_G(X)$ $(p \in P)$ form a $\mathbf{Z}$-basis of $K_G(X)$. Thus $K_G(X)$ may be identified with the group ring $\mathbf{Z}[P]$. It is also known that under this identification the canonical ring homomorphism $R_G \to K_G(X)$ becomes the inclusion of the $W$-invariants $\mathbf{Z}[P]^W$ into $\mathbf{Z}[P]$. It follows that we may identify $K_M(X)$ with $\mathcal{R}_0$ = group ring of $P$ over $\mathbf{Z}[q, q^{-1}]$ and the canonical ring homomorphism $R_M \to K_M(X)$ with the inclusion $\mathcal{R}_0^W \to \mathcal{R}_0$. We shall denote by $\mathcal{R}$ the quotient field of $\mathcal{R}_0$. With these identifications, the map $T_s \colon K_M(X) \to K_M(X)$ becomes the $\mathbf{Z}[q, q^{-1}]$-linear map $T_s \colon \mathcal{R}_0 \to \mathcal{R}_0$ given by

$$T_s(\lambda) = \frac{\lambda - s(\lambda)}{\alpha_s - 1} - q\frac{\lambda - s(\lambda)\alpha_s}{\alpha_s - 1} \qquad (\lambda \in P). \tag{8.1}$$

Note that this is a priori an element of $\mathcal{R}$. But it is easily seen that it is actually in $\mathcal{R}_0$. (If we specialize $q$ to 1 this is just the action $\lambda \to s(\lambda)$ of $W$ on $P$. If we specialize $q$ to 0 we obtain essentially Demazure's operator [**1**]. Thus our operator (8.1) is the simplest possible interpolation between these two special cases.)

The map $\theta_p\colon K_M(X) \to K_M(X)$ becomes with the previous identifications the $Z[q,q^{-1}]$-linear map $\theta_p\colon \mathcal{R}_0 \to \mathcal{R}_0$ defined by

$$\theta_p(\lambda) = \lambda p^{-1}. \tag{8.2}$$

We must show that the endomorphisms (8.1), (8.2) of $\mathcal{R}_0$ verify the identities (5.1)–(5.5).

The identities (5.1), (5.3) and (5.4) are immediate. Now let $s \neq t$ be two simple reflections in $S$ and let $\langle s,t\rangle$ be the subgroup of $W$ they generate. Let $\mathbf{\Phi}_{s,t}$ be the set of roots which are $\mathbf{Z}$-combinations of the simple roots $\alpha_s, \alpha_t$ and let $\mathbf{\Phi}^+_{s,t} \cup \mathbf{\Phi}^-_{s,t}$ be its partition into positive and negative roots. Let $\rho_{s,t}$ be the element of $P$ such that

$$\rho^2_{s,t} = \prod_{\alpha\in\mathbf{\Phi}^+_{s,t}} \alpha.$$

Let

$$\Psi^+ = \prod_{\alpha\in\mathbf{\Phi}^+_{s,t}} (1-q\alpha), \qquad \Psi^- = \prod_{\alpha\in\mathbf{\Phi}^-_{s,t}} (1-q\alpha).$$

For any element $\xi \in \mathcal{R}_0$ we define $\operatorname{Alt}_{s,t}(\xi) = \sum_{w\in\langle s,t\rangle}(-1)^{l(w)}w(\xi)$, where $l$ is the length function on $W$.

We denote the product $T_sT_tT_s\cdots$ ($i$ factors) by $T_{s,i}$ and similarly we denote the product $T_tT_sT_t\cdots$ ($i$ factors) by $T_{t,i}$. With these notations we state the following identity which is verified by direct computation. (Here $m$ = order of $st$ in $W$.)

$$\begin{aligned}&\left(1+\sum_{1\le i\le m} T_{s,i} + \sum_{1\le i\le m-1} T_{t,i}\right)(\lambda)\\ &\quad= \operatorname{Alt}_{s,t}(\lambda\rho_{s,t}\Psi^-)\cdot(\operatorname{Alt}_{s,t}(\rho_{s,t}))^{-1} \qquad (\lambda\in P).\end{aligned} \tag{8.3}$$

The right-hand side of this identity is symmetric in $s,t$. Hence so is the left-hand side. It follows that $T_{s,m} = T_{t,m}$ and (5.2) follows. (Alternatively, one can use the following identity:

$$\begin{aligned}&\left((-q)^m + \sum_{1\le i\le m}(-q)^{m-i}T_{s,i} + \sum_{1\le i\le m-1}(-q)^{m-i}T_{t,i}\right)(\lambda)\\ &\quad= \rho^{-1}_{s,t}\Psi^+\operatorname{Alt}_{s,t}(\lambda)\cdot(\operatorname{Alt}_{s,t}(\rho_{s,t}))^{-1} \qquad (\lambda\in P).)\end{aligned}$$

With the assumption of (5.5) we have for $\lambda \in P$:

$$\begin{aligned}T_s\theta_{s(p)}T_s(\lambda) &= T_s\frac{\lambda - s(\lambda) - q(\lambda - s(\lambda)\alpha_s)}{(\alpha_s-1)s(p)}\\ &= \frac{1}{\alpha_s - 1}\left(\frac{\lambda - s(\lambda) - q(\lambda - s(\lambda)\alpha_s)}{(\alpha_s-1)p\alpha_s^{-1}}(1-q)\right.\\ &\qquad\qquad\left. - \frac{s(\lambda) - \lambda - q(s(\lambda) - \lambda\alpha_s^{-1})}{(\alpha_s^{-1}-1)p}(1-q\alpha_s)\right)\\ &= q\lambda p^{-1} = q\theta_p(\lambda).\end{aligned}$$

We now show that the operators $T_s, \theta_p: \mathcal{R}_0 \to \mathcal{R}_0$ are $\mathcal{R}_0^W$-linear. This is trivial for $\theta_p$. For $T_s$, we have the following multiplicative property (whose proof is trivial):

$$T_s(f \cdot g) = (T_s f) \cdot g + s(f) \cdot T_s(g) - qs(f) \cdot g$$

for any $f, g \in \mathcal{R}_0$. If $f = s(f)$, then clearly $T_s f = qf$ so that $T_s(f \cdot g) = f \cdot T_s(g)$. In particular, if $f \in \mathcal{R}_0^W$, then $T_s(f \cdot g) = f \cdot T_s(g)$ for all $g \in \mathcal{R}_0$ and all $s \in S$. This shows that $T_s$ is $\mathcal{R}_0^W$-linear.

**9. Remarks.** (a) By a theorem of Pittie, $\mathcal{R}_0$ is a free $\mathcal{R}_0^W$-module of rank $|W|$.

(b) The formula (8.3) has the following interpretation. Let $P_{s,t}$ be the parabolic subgroup $\bigcup_{w \in \langle s,t \rangle} BwB$ and let $\pi_{s,t}$ be the natural map $G/B \to G/P_{s,t}$. Let us define $T_w$ for $w \in W$ as $T_{s_1} T_{s_2} \cdots T_{s_r}$, where $s_1 s_2 \cdots s_r$ is a reduced expression for $W$. Then for any $M$-equivariant algebraic vector bundle $E$ on $X = G/B$, we have

$$\sum_{w \in \langle s,t \rangle} T_w E = \sum_i (-1)^i \pi_{s,t}^* (\pi_{s,t})_* (E \otimes \Omega_{s,t}^i),$$

where $\Omega_{s,t}^i$ is the vector bundle on $X$ of holomorphic differential $i$-forms along the fibres of $\pi_{s,t}$ regarded as an $M$-equivariant bundle with the obvious action of $G$ and with the action of $\mathbf{C}^*$ given by scalar multiplication by $z^i$ on each fibre. (Compare (4.2).)

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