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REMARKS ON THE HOMOTOPY TYPE OF GROUPS OF SYMPLECTIC DIFFEOMORPHISMS

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ABSTRACT. Let (X, ω) be a symplectic manifold without boundary, $G(X)$ the identity component of its group of compactly supported diffeomorphisms, and $H_\omega(X)$ the subgroup of $G(X)$ consisting of all symplectic diffeomorphisms. In this note, we give examples in which $H_\omega(X)$ is not homotopy equivalent to $G(X)$.

Let (X, ω) be a symplectic manifold without boundary, $G(X)$ the identity component of the group of all its compactly supported diffeomorphisms, and $H_\omega(X)$ the subgroup of $G(X)$ consisting of all symplectic diffeomorphisms. By Moser [**3**], there is a fibration

$$H_\omega(X) \to G(X) \to \operatorname{Symp}\omega,$$

where $\operatorname{Symp}\omega$ is the orbit $\{g^*\omega : g \in G(X)\}$ of $G(X)$ in the space of symplectic forms on X which equal ω near infinity. Very little is known about the structure of the space $\operatorname{Symp}\omega$, and hence about the relation of $H_\omega(X)$ to $G(X)$. In this note, we give examples in which $\operatorname{Symp}\omega$ is not contractible, thereby proving that $H_\omega(X)$ is not always homotopy equivalent to $G(X)$. For example, we prove

PROPOSITION 1. *Let $X = \mathbf{C}P^2 \# \overline{\mathbf{C}P}^2$ be the Kähler manifold obtained by blowing up a point of $\mathbf{C}P^2$, with Kähler form ω. Then $\pi_1 H_\omega(X)$ does not surject onto $\pi_1 G(X)$.*

PROPOSITION 2. *Let $X = \mathbf{C}^2 - \{0\}$ with its usual symplectic form. Then $H_\omega(X)$ is not connected.*

Our methods of proof are very elementary. For example, we prove Proposition 1 by using obstruction theory to show that there is a loop in $\operatorname{Symp}\omega$ which does not contract in the space $\mathcal{S}(X)$ of all nondegenerate 2-forms on X which equal ω outside some compact set. Other examples with $H_\omega(X) \not\simeq G(X)$ are discussed in [**2**].

For simplicity, let us begin by considering the parallelizable manifold $X = T^2 \times S^2$ with its usual Kähler form. Let $f_t = \mathrm{id} \times g_t$, where g_t is a rotation of S^2 by $2\pi t$ about some fixed axis, and let x_0 be a fixed point of f_t. Because the loop $df_{2t}(x_0)$, $0 \le t \le 1$, contracts in SO(4), we may modify the f_t near x_0 so that they are the identity near x_0. To be precise, let $N \cong D^4$ be a neighbourhood of x_0 on which f_t acts as a rotation. Choose a smooth family $k_{s,t}: 0 \le s, t \le 1$ in SO(4) such that

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$k_{s,t} = \text{id}$ if $t = 0, 1$ or if $s < \frac{1}{4}$, and such that $k_{s,t} = f_{2t}$ for $s > \frac{1}{2}$. Then define h_t by

$$h_t = f_{2t} \quad \text{outside } N, \text{ and}$$
$$h_t(x) = k_{\|x\|,t}(x) \quad \text{for } x \in N \equiv D^4.$$

Clearly h_t, $0 \leq t \leq 1$, is a loop in $G(X - x_0)$.

PROPOSITION 3. *The image of* $\{h_t\}$ *in* $\mathcal{S}(X - x_0)$ *is not contractible. Hence* $\pi_1 H_\omega(X - x_0)$ *does not surject onto* $\pi_1 G(X - x_0)$.

PROOF. Let J_0 be the standard (almost) complex structure on X, and denote by $\mathcal{J}_0$ the space of all almost complex structures on X which equal J_0 outside some compact set. Then $\mathcal{J}_0 \simeq \mathcal{S}(X - x_0)$. Also $G(X - x_0)$ acts on $\mathcal{J}_0$ by

$$(h \cdot J)_{hx} = dh_x \circ J_x \circ (dh_x)^{-1}.$$

Clearly, it will suffice to show that the loop $\{h_t \cdot J_0\}$ does not contract in $\mathcal{J}_0$.

Because X is parallelizable, the space J of all almost complex structures on X is homotopy equivalent to the space $\mathcal{M}(X, S^2)$ of smooth maps from X to $S^2 \cong \mathrm{SO}(4)/U(2)$. To make this explicit, identify X with $S^1 \times (\mathbf{R}^3 - 0)/\mathbf{Z}$ and trivialize TX using the obvious trivialization of $T(S^1 \times \mathbf{R}^3)$. Then define $F: J \to \mathcal{M}(X, S^2)$ by

$$F(J)(x) = \text{unit vector along } \mathrm{pr}_x(J_x(\partial/\partial\theta)),$$

where pr_x is the projection of $T_x X$ onto the tangent space to the $\mathbf{R}^3$-factor, $\partial/\partial\theta$ is a vector tangent to the S^1-factor, and where one uses any metric. Because $J_0(\partial/\partial\theta)$ points along the radial direction in $\mathbf{R}^3$, $F(J_0)$ is the projection map of $X = T^2 \times S^2$ to S^2. Hence $\mathcal{J}_0$ is homotopy equivalent to the subspace $\mathcal{M}_0$ of $\mathcal{M}(X, S^2)$ consisting of maps which equal the projection at x_0.

The loop $\{h_t \cdot J_0\}$ in $\mathcal{J}_0$ gives rise to a loop in $\mathcal{M}_0$ and hence to a map $H: X \times S^1 \to S^2$. The following lemmas imply that H does not extend to a continuous map $X \times D^2 \to S^2$ which takes $x_0 \times D^2$ to z_0, where $x_0 = (y_0, z_0) \in T^2 \times S^2 = X$. Proposition 3 clearly follows.

Let β be a 2-cell in X which is disjoint from N and has the form $y_1 \times S^2$, and let γ be a 1-cell $\gamma' \times z_0$, where γ' is a path in T^2 from y_0 to y_1. Thus the endpoints of γ are x_0 and $x_1 = (y_1, z_0)$.

LEMMA 1. *If* $\tilde{H}: \gamma \times D^2 \to S^2$ *is any extension of* H *such that* $\tilde{H}(x_0 \times D^2) = z_0$, *then* $\tilde{H}: (x_1 \times D^2, x_1 \times S^1) \to (S^2, z_0)$ *represents a nonzero element of* $\pi_2 S^2$.

LEMMA 2. *There is a nonzero obstruction to extending such* $\tilde{H}$ *over* $\beta \times D^2$.

PROOF OF LEMMA 1. It suffices to show that the map $H: (\gamma \times S^1, \partial\gamma \times S^1) \to (S^2, z_0)$ has nonzero degree. Since h_t preserves J_0 outside N, $H(x, t)$ equals the projection of x onto S^2 for $x \notin N$, $t \in S^1$. In particular, $H(x, t) = z_0$ for $x \in \gamma - N$, and we need only consider the restriction of H to $\gamma_0 \times S^1$, where $\gamma_0 = \gamma \cap N$.

To simplify the calculation, we assume as we may that γ_0 is tangent to the vector field $J_0(\partial/\partial\theta)$, and that N is identified with the unit disk D^4 in $\mathbf{R}^4$ in such a way that $\partial/\partial\theta$ points along the x_1-direction, γ_0 lies along the x_2-axis, and f_t rotates in the x_3, x_4-plane. Further, we choose the rotations $k_{s,t}$ to fix the x_1-axis, so that they are in SO(3). Then $h_t(N_0) = N_0$, where N_0 is the 3-disk $\{x_1 = 0\} \cap D^4$. Now recall that H is given by

$$(x, t) \mapsto F(h_t \cdot J_0)(x) = \text{unit vector along } \mathrm{pr}_x(dh_t \circ J_0 \circ dh_t^{-1}(\partial/\partial\theta)).$$

Since the vector fields $\partial/\partial\theta$ and $dh_t^{-1}(\partial/\partial\theta)$ are transverse to N_0 and equal along ∂N_0, they are homotopic on $\gamma_0 \operatorname{rel} \partial\gamma_0$. The above remarks then imply that $H|\gamma_0 \times S^1$ is homotopic $\operatorname{rel} \partial\gamma_0 \times S^1$ to the map which takes (x, t) to the unit tangent to $h_t(\gamma_0)$ at $h_t(x)$. But this tangent always makes an acute angle with the outward unit radial vector $v(x, t)$ at $h_t(x) \in N_0 \equiv D^3$. (Here we define $v(x_0, t) = v_0$ to be the unit tangent to γ_0 at x_0.) Hence H is homotopic $\operatorname{rel} \partial\gamma_0 \times S^1$ to the map which takes (x, t) to $v(x, t)$. Using the fact that $v(x, t) = k_{\|x\|, t}(v_0)$, one easily sees that if H had zero degree, the loop $\{f_{2t}\}$ would contract in SO(2). □

PROOF OF LEMMA 2. The obstruction to extending $\tilde{H}$ over $\beta \times D^2$ is the map

$$\chi: S^3 = \partial(D^2 \times D^2) = (\partial D^2 \times D^2) \cup (D^2 \times \partial D^2) \to S^2,$$

where $\chi|\partial D^2 \times D^2$ and $\chi|D^2 \times \partial D^2$ are given by

$$\partial D^2 \times D^2 \to x_1 \times D^2/\partial D^2 \xrightarrow{\varphi} S^2$$

and

$$D^2 \times \partial D^2 \to \beta \times \partial D^2 \xrightarrow{\mathrm{pr}} \beta \xrightarrow{\psi} S^2.$$

Here φ is induced by H, and ψ is the map $F(J_0)$ and so has degree 1. We may assume that χ is smooth. Then, because φ has nonzero degree by Lemma 1, the inverse images under χ of distinct regular points are linked. By Hopf's classification of maps $S^3 \to S^2$, this implies that χ is essential. □

Next, we prove an analogous result for the Kähler manifold $X = \mathbf{C}P^2$. Let $x_0 = [1:0:0]$ and define $f_t: X \to X$ by

$$f_t[z_0 : z_1 : z_2] = [z_0 : z_1 : e^{2\pi i t} z_2].$$

Then f_t fixes x_0, and, just as before, the loop $\{f_{2t}\}$ may be modified in a neighborhood N of x_0 to yield a loop $\{h_t\}$ in $G(X - x_0)$.

PROPOSITION 4. *The image of* $\{h_t\}$ *in* $\mathcal{S}_0(X - x_0)$ *is not contractible.*

PROOF. This is much the same as before, except that we must deal with the twisting of TX. Let γ be the 1-cell $[\lambda : 1 - \lambda : 0]$, $0 \le \lambda \le 1$, in X, and let β be the 2-cell $\{z_0 = 0\}$ with vertex $x_1 = [0:1:0]$. As before, we assume that N does not meet β. The space $\mathcal{S}$ of nondegenerate 2-forms on X is homotopy equivalent to the space $\mathcal{M}$ of sections of a certain S^2-bundle $E \to X$, and our loop corresponds to a section H of the pull-back bundle $E \times S^1 \to X \times S^1$. We will show that H does not extend to a section of $E \times D^2 \to X \times D^2$ which equals the base section s_0 over $x_0 \times D^2$, where s_0 corresponds to the standard almost complex structure J_0.

Because the proof of Lemma 1 just involves properties of H on $N \times S^1$, it applies in the present situation. Hence we have

LEMMA 3. *If* $\tilde{H}$ *is any extension of* H *over* $\gamma \times D^2$ *which equals* s_0 *on* $x_0 \times D^2$, *then the map* $\tilde{H}: (x_1 \times D^2, x_1 \times S^1) \to (F, *)$ *represents a nonzero element of* $\pi_2 F$, *where* F *is the fiber of* $E \to X$ *at* x_1.

Next, observe that $E \to X$ is an oriented S^2-bundle with distinguished section s_0, and so is the suspension of an oriented S^1-bundle ξ over X. Thus it is determined by the Euler class $e(\xi)$ of ξ. We claim that $e(\xi) = c_1(X)$. To see this, note that any $U(2)$-bundle η has an associated oriented bundle with fiber $SO(4)/U(2) \cong S^2$. Since the latter has a distinguished section, it determines an oriented circle bundle ξ_η. It is not hard to see that $e(\xi_\eta) = mc_1(\eta)$, for some integer m which is independent of η. But we saw in the proof of Proposition 3 that, when η is the tangent bundle to $T^2 \times S^2$, then ξ_η restricts on pt $\times S^2$ to the unit tangent bundle of S^2. Hence $m = 1$ as claimed.

Now identify the 2-cell $\beta \cong S^2$ with $D^+ \cup D^-$, where D^+ and D^- are 2-disks which meet in a circle $D^+ \cap D^-$ which contains x_1. Trivialize E over D^+ and D^- so that the section s_0 is the constant map $D^\pm \to * \in S^2$. Then the attaching map $A: \partial D^- \times S^2 \to \partial D^+ \times S^2$ which defines $E|\beta$ is $(t,z) \mapsto (t, g_{3t}z)$, where g_t is a rotation through $2\pi t$ about the axis through $*$. We have a section $\tilde{H}$ of $E \times D^2 \to X \times D^2$ which is defined over $\beta \times S^1 \cup \{x_1\} \times D^2$. By Lemma 3, the restriction of $\tilde{H}$ to $\{x_1\} \times D^2$ defines a nontrivial element of $\pi_2 S^2$. Further, since the h_t preserve J_0 on β, the section $\tilde{H}$ is the pull-back of the distinguished section s_0 over $\beta \times S^1$. Any extension φ of $\tilde{H}$ over $\partial D^- \times D^2$ is homotopic to one of the maps

$$\varphi_k(t,x) = g_{kt}(\alpha(x)), \qquad k \in \mathbf{Z},$$

where $\alpha = \tilde{H}|x_1 \times D^2$, and we use the above described trivialization of $E|D^-$. As in Lemma 2, one sees that φ extends over $D^- \times D^2$ if and only if $\varphi \sim \varphi_0$. But, in the trivialization of $E|D^+$, φ_0 is the map $(t,x) \mapsto g_{3t}\alpha(x)$. Hence $\tilde{H}$ cannot extend over both $D^- \times D^2$ and $D^+ \times D^2$. □

PROOF OF PROPOSITION 1. Let us think of $X = \mathbf{C}P^2 \# \overline{\mathbf{C}P}^2$ as the manifold $\mathbf{C}P^2$ of Proposition 4 with a little disk around x_0 replaced by a copy of $\overline{\mathbf{C}P}^2$ − disk, and let h_t be the extension by the identity of the diffeomorphism h_t of that proposition. We must show that the image in $\mathcal{S}$ of the loop $\{h_t\}$ is not contractible. Now, X has a cell decomposition with 2-skeleton $S^2 \cup I \cup S^2 = \beta \cup \gamma \cup \overline{\beta}$, where β and γ are essentially as before and $\overline{\beta} \subset \overline{\mathbf{C}P}^2$ − disk. The h_t define a section H over $X \times S^1$ of an S^2-bundle $E \times D^2 \to X \times D^2$, which equals the pull-back of the distinguished section on $(\beta \cup \overline{\beta}) \cup S^1$. Let $\tilde{H}$ be some extension of H to $\gamma \times D^2$, and let $x_1, \overline{x}_1$ be the endpoints of γ. Then the proof of Proposition 4 shows that $\tilde{H}$ cannot be extended over $\beta \times D^2$ unless $\tilde{H}|x_1 \times D^2$ represents the zero element of $\pi_2 S^2$. Similarly, $\tilde{H}$ cannot be extended over $\overline{\beta} \times D^2$ unless $\tilde{H}|\overline{x}_1 \times D^2$ represents the zero element of $\pi_2 S^2$. But the proof of Lemma 1 shows that at least one of these elements must be nonzero. □

Note. Similar results may be proved for many other manifolds, and in particular for $\mathbf{C}P^n \# \overline{\mathbf{C}P}^n$.

PROOF OF PROPOSITION 2. Let $X = \mathbf{C}^2 - \{0\}$ with the obvious symplectic form, and let $W = \{x: \|x\| > 3\} \cup \{x: |z_2| < 1\} \subset \mathbf{C}^2$, where $x = (z_1, z_2)$. Let f_t be the rotation of $\mathbf{C}^2$ through $2\pi t$ in the z_1-plane, and choose a smooth function $s: [0,3] \to [0,2]$ which equals 2 on $[0,1]$ and 0 on $[2,3]$. Then define $g_t: W \to W$, $0 \le t \le 1$, by

$$g_t(x) = \begin{cases} f_{s(|z_1|)t}(x) & \text{if } |z_2| < 1, \\ x & \text{if } \|x\| > 3. \end{cases}$$

Then g_t is a symplectic isotopy of W, which extends to a compactly supported symplectic isotopy $\tilde{g}_t$ of $\mathbf{C}^2$ by [**1**, II.2.1]. Further, $\tilde{g}_1$ equals the identity near 0 and therefore may be considered as a compactly supported diffeomorphism of X. We claim that $\tilde{g}_1$ is in the identity component of $G(X)$ but not in that of $H_\omega(X)$. To see this, let $\mathrm{Diff}^c(\mathbf{C}^2, 0)$, resp. $\mathrm{Diff}^c X$, be the group of compactly supported diffeomorphisms of $\mathbf{C}^2$ which fix 0, resp. which are the identity near 0, and consider the fibration

$$\mathrm{Diff}^c X \to \mathrm{Diff}^c(\mathbf{C}^2, 0) \xrightarrow{\nu} \mathrm{GL}^+(4, \mathbf{R}),$$

where $\nu(g) = dg_0$. Since the loop $\{\nu(\tilde{g}_t)\}$ is contractible, $\tilde{g}_1$ belongs to the identity component $G(X)$ of $\mathrm{Diff}^c X$. Thus $\tilde{g}_1 \in H_\omega(X)$. However, there is an obstruction to the existence of an isotopy in $H_\omega(X)$ from $\tilde{g}_1$ to the identity. For, if one defines a map ψ from $H_\omega(X)$ to the loop space $\Omega\,\mathrm{Sp}(4, \mathbf{R})$ by

$$\psi h(s) = dh_{(s,0,0,0)}, \qquad s > 0,$$

then one easily sees that $\psi\tilde{g}_1$ represents a nontrivial element of $\pi_0(\Omega\,\mathrm{Sp}(4, \mathbf{R})) = \mathbf{Z}$. $\square$

Note. A similar result holds for $\mathbf{C}^n - \{0\}$. More generally, if Y is a connected noncompact symplectic manifold, then the diffeomorphism $\tilde{g}_1$ may be transferred to Y to give an element of $H_\omega(Y - \mathrm{pt})$ which will not be in the identity component if $c_1 Y = 0$. It is also not hard to construct a noncompact manifold X with one end such that $H_\omega(X)$ is not connected. However, I have not been able to find a compact example.

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