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FUNCTIONS WITH A DENSE SET OF PROPER LOCAL MAXIMUM POINTS

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ABSTRACT. Let X be any metric space. The existence of continuous real functions on X, with a dense set of proper local maximum points, is shown. Indeed, given any σ-discrete set $S \subset X$, the set of all $f \in C(X)$, which assume a proper local maximum at each point of S, is a dense subset of $C(X)$. This implies, for a perfect metric space X, the density in $C(X, Y)$ of "nowhere constant" continuous functions from X to a normed space Y. In this way, two questions raised in [2] are solved.

The existence of continuous functions $f\colon \mathbf{R} \to \mathbf{R}$ having some proper local maximum point within each open subset of $\mathbf{R}$ is well known. A nice construction of such a function is given in [1]. In this note we show that continuous real functions with this property do exist on any metric space X. Indeed, we prove (Theorem 1) that for every σ-discrete set $S \subset X$, the set of continuous real functions on X, which have a proper local maximum at each point of S, is dense in $C(X)$, endowed with a certain topology which, in general, is strictly finer than that of uniform convergence; in particular, functions with a dense set of proper local maximum points are dense in $C(X)$. As a corollary, if X is a perfect metric space, we get the density in $C(X, Y)$ of *nowhere constant* continuous functions from X to a normed space Y. This answers two questions recently raised in [2] and enables us to improve some results established there.

Throughout, X is a metric space with metric d, Y is a normed space with norm $\| \ \|$, and $C(X, Y)$ denotes the set of all continuous functions $f\colon X \to Y$. When $Y = \mathbf{R}$ we put $C(X) = C(X, Y)$. We denote by τ the topology on $C(X, Y)$ in which basic neighbourhoods of $f \in C(X, Y)$ are the sets

$$\{ g \in C(X, Y) \mid \|g(x) - f(x)\| < \varepsilon(x) \text{ for each } x \in X \}$$

with $\varepsilon \in C(X)$, $\varepsilon > 0$ everywhere in X. It is clear that τ is stronger than the topology υ induced by the metric ρ of uniform convergence, i.e.,

$$\rho(f, g) = \min\{1, \sup\{ \|f(x) - g(x)\| \mid x \in X \}\}.$$

Moreover, it is easy to realize that τ and υ coincide if and only if X is compact.

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For subsets of X the term *discrete* is used here in the following sense: a set $D \subset X$ is said to be discrete if D has no accumulation points. A σ-discrete set is a countable union of discrete sets. For collections $\mathscr{F}$ of subsets of X our terminology is standard: $\mathscr{F}$ is said to be discrete if every point $x \in X$ has a neighbourhood meeting at most one set in $\mathscr{F}$. A σ-discrete collection is a countable union of discrete collections.

Since the definition of discrete set given here is nonstandard, some comments are necessary. Obviously a discrete set as defined here is also discrete in the usual sense, i.e., its relative topology is the discrete topology. The converse is not true (e.g., $X = \mathbf{R}$, $D = \{1/n | n = 1, 2, \dots\}$). Nevertheless, if a set D is discrete in the usual sense, then it is also σ-discrete according to our definition. Indeed, for each $\rho > 0$, the set $D_\rho = \{x \in D | d(x, D - \{x\}) \geqslant \rho\}$ has no accumulation points; hence $D = \bigcup_{\rho > 0} D_\rho = \bigcup_{n=1}^{\infty} D_{1/n}$ is σ-discrete in our sense. It follows that, in a metric space, the definition of σ-discrete set given here and the usual one are equivalent. This is no longer true in a general topological space. We do not give any explicit proof of the last remark. However, a counterexample can easily be constructed by the interested reader in the space of all real functions on $\mathbf{R}$ endowed with the topology of pointwise convergence.

Given a function $f\colon X \to \mathbf{R}$, we say that $x \in X$ is a proper local maximum point for f if $f(U - \{x\}) \subset (-\infty, f(x))$ for some neighbourhood U of x. The set of all proper local maximum points for f is denoted by $M(f)$. Note that for every $t > 0$, the set

$$M_t(f) := \{x \in X | 0 < d(z, x) < t \Rightarrow f(z) < f(x)\}$$

is discrete; hence, the set

$$M(f) = \bigcup_{t > 0} M_t(f) = \bigcup_{n=1}^{\infty} M_{1/n}(f)$$

is σ-discrete. Conversely, Theorem 1 shows that for any σ-discrete set $S \subset X$ there is always a continuous function $f\colon X \to \mathbf{R}$ such that $M(f) \supset S$.

For the reader's convenience we state a result from **[2]** that will be used in the sequel. By $\mathscr{R}(X, Y)$ we denote the set of all $f \in C(X, Y)$ which are *nowhere constant* (locally nonconstant according to the terminology of **[2]**), i.e., such that $\operatorname{int} f^{-1}(y) = \varnothing$ for all $y \in Y$. When $Y = \mathbf{R}$ we put $\mathscr{R}(X) = \mathscr{R}(X, Y)$. The convex hull of a set $W \subset Y$ is denoted by $\operatorname{conv}(W)$.

THEOREM A (**[2**, THEOREM 2.1]). *Suppose X is locally connected and $\mathscr{R}(X) = \varnothing$. Then, for every function $f \in C(X, Y)$ and every positive constant ε, there exists a function $g \in \mathscr{R}(X, Y)$ such that $\rho(g, f) \leqslant \varepsilon$. Moreover,* $g(X) \subset \operatorname{conv}(f(X))$ *provided that f is not constant.*

To begin, we prove two lemmas.

LEMMA 1. *Let $D \subset X$ be a nonempty discrete set. Then there exists a discrete collection $\{B_s | s \in D\}$ of closed balls with each B_s centered at s.*

Proof. If D is a finite set, then any collection of pairwise disjoint closed balls, with centers at points of D, is discrete. This is no longer true, in general, if the set D is infinite. Then we proceed as follows.

For each $x \in X$ let d_x denote the positive number $d(x, D - \{x\})$. Also, fix any $\gamma \in (0, \frac{1}{2})$, and denote by B_s, $s \in D$, the closed ball centered at s with radius $r_s := \gamma d_s$. Then $\{B_s | s \in D\}$ is a discrete collection. Indeed, we claim that for every $x \in X$, any ball C_x with center at x and radius $\rho_x \leqslant (\frac{1}{2} - \gamma) d_x$ meets at most one ball B_s. This is proved by contradiction as follows. Assume that $C_x \cap B_{s_1}$ and $C_x \cap B_{s_2}$ are nonempty for some $x \in X$, $s_1, s_2 \in D$, $s_1 \neq s_2$, and let $z_i \in C_x \cap B_{s_i}$, $i = 1, 2$. Then $d_x \geqslant \max\{d_{s_1}, d_{s_2}\}$, for otherwise, assuming for instance $d_{s_1} > d_x$, $d_{s_1} \geqslant d_{s_2}$, we would get the contradiction

$$\begin{aligned} d_{s_1} &\leqslant d(s_1, s_2) \leqslant d(s_1, z_1) + d(z_1, z_2) + d(z_2, s_2) \\ &\leqslant r_{s_1} + 2\rho_x + r_{s_2} \leqslant \gamma d_{s_1} + (1 - 2\gamma) d_x + \gamma d_{s_2} \\ &< \gamma d_{s_1} + (1 - 2\gamma) d_{s_1} + \gamma d_{s_1} = d_{s_1}. \end{aligned}$$

The contradiction

$$\begin{aligned} d_x &\leqslant d(x, s_1) \leqslant d(x, z_1) + d(z_1, s_1) \leqslant \rho_x + r_{s_1} \\ &\leqslant (\tfrac{1}{2} - \gamma) d_x + \gamma d_{s_1} \leqslant (\tfrac{1}{2} - \gamma) d_x + \gamma d_x = d_x/2 \end{aligned}$$

follows, assuming, for instance, $s_1 \neq x$, and the lemma is proved. □

Lemma 2. *Let $\varphi, \eta \in C(X)$ with $\eta > 0$ everywhere in X. Also, let $D \subset X$ be a nonempty discrete set and H a closed set with $D \cap H = \varnothing$. Then there exist $\psi \in C(X)$ and $\{B_s | s \in D\}$, a discrete collection of closed balls with each B_s centered at s, such that*

(i) $(\bigcup_{s \in D} B_s) \cap H = \varnothing$,
(ii) $\psi = \varphi$ *in* $X - \bigcup_{s \in D} B_s$,
(iii) $\varphi \leqslant \psi < \varphi + \eta$ *everywhere in* X, *and*
(iv) $\psi(x) < \psi(s)$ *for every* $s \in D$ *and* $x \in B_s - \{s\}$.

Proof. By Lemma 1 it is possible to associate a positive number r_s with each $s \in D$ in such a way that the collection of balls B_s, with center at $s \in D$ and radius r_s, is discrete. Also, it is clearly possible, decreasing the numbers r_s if necessary, to fulfill condition (i). In the same way, owing to the continuity of φ and η, it can be assumed that, for every $s \in D$, the inequalities

$$(*) \qquad \varphi(x) < \varphi(s) + \tfrac{1}{2}\eta(s) < \varphi(x) + \eta(x)$$

hold at each point x of B_s.

Now define $\psi: X \to \mathbf{R}$ as follows:

$$\psi(x) = \varphi(x) \quad \text{if } x \in X - \bigcup_{s \in D} B_s,$$

$$\psi(x) = \big(1 - r_s^{-1} d(x, s)\big)\big(\varphi(s) + \tfrac{1}{2}\eta(s)\big) + r_s^{-1} d(x, s)\varphi(x) \quad \text{if } x \in B_s, s \in D.$$

Then, by the discreteness of $\{B_s | s \in D\}$, ψ is well defined and continuous. Also, by definition, (ii) is satisfied. Finally, having in mind the inequalities $(*)$, it is an easy matter to check the validity of conditions (iii) and (iv). The lemma is therefore proved. □

Now we prove our main theorem.

THEOREM 1. *Let $f, \varepsilon \in C(X)$ with $\varepsilon > 0$ everywhere in X. Also let $S \subset X$ be a nonempty σ-discrete set and $K \subset X$ a closed set with $S \cap K = \varnothing$. Then there exists $g \in C(X)$ such that*

(i) $g|_K = f|_K$,

(ii) $f \leqslant g < f + \varepsilon$ *everywhere in X, and*

(iii) $M(g) \supset S$.

PROOF. If S is a discrete set, the existence of such a function g is guaranteed by Lemma 2. Hence, assume that S is not discrete. Then we have $S = \bigcup_{n=0}^{\infty} D_n$, with D_0, $D_1, \ldots$ pairwise disjoint nonempty discrete sets. We set $E_n = \bigcup_{k=0}^{n} D_k$, $n = 0, 1, \ldots$.

We start by applying Lemma 2 with $\varphi = f$, $\eta = \varepsilon/2$, $D = D_0$, and $H = K$. Accordingly, there exist $f_0 \in C(X)$ and $\{B_s | s \in D_0\}$, a collection of closed balls with each B_s centered at s, such that

$(1)_0$ $\{B_s | s \in D_0\}$ is a discrete collection of sets,

$(2)_0$ $(\bigcup_{s \in D_0} B_s) \cap K = \varnothing$,

$(3)_0$ $f = f_0$ in $X - \bigcup_{s \in D_0} B_s$,

$(4)_0$ $f \leqslant f_0 < f + \varepsilon/2$ everywhere in X,

$(5)_0$ $f_0(x) < f_0(s)$ for every $s \in D_0$ and $x \in B_s - \{s\}$.

Next we show by induction that there exist $\{f_n | n = 1, 2, \ldots\}$, a sequence in $C(X)$, and $\{B_s | s \in \bigcup_{n=1}^{\infty} D_n\}$, a collection of closed balls with B_s centered at s, such that the following conditions are satisfied for every $n = 1, 2, \ldots$:

$(1)_n$ $\{B_s | s \in D_n\}$ is a discrete collection of sets,

$(2)_n$ $(\bigcup_{s \in D_n} B_s) \cap (K \cup E_{n-1}) = \varnothing$,

$(3)_n$ $f_n = f_{n-1}$ in $X - \bigcup_{s \in D_n} B_s$,

$(4)_n$ $f_{n-1} \leqslant f_n < f_{n-1} + \varepsilon/2^{n+1}$ everywhere in X,

$(5)_n$ $f_n(x) < f_n(s)$ for every $s \in E_n$ and $x \in B_s - \{s\}$,

$(6)_n$ $s \in D_n$, $t \in E_{n-1}$, $B_s \cap B_t \neq \varnothing$ imply $s \in B_t$.

To prove this, let us assume that functions $f_0, \ldots, f_n$ and balls $\{B_s | s \in E_n\}$ have been found in such a way that conditions $(1)_k$–$(5)_k$, $k = 0, \ldots, n$, and, if $n > 0$, also $(6)_k$, $k = 1, \ldots, n$, are satisfied, and construct f_{n+1} and $\{B_s | s \in D_{n+1}\}$ such that $(1)_{n+1}$–$(6)_{n+1}$ hold.

Decompose D_{n+1} as $D_{n+1} = D'_{n+1} \cup D''_{n+1}$, with $D'_{n+1} = D_{n+1} - \bigcup_{t \in E_n} B_t$ and $D''_{n+1} = D_{n+1} - D'_{n+1}$.

If $D'_{n+1} = \varnothing$ we introduce h by setting $h = f_n$.

If $D'_{n+1} \neq \varnothing$ we apply Lemma 2 again, with $\varphi = f_n$, $\eta = \varepsilon/2^{n+3}$, $D = D'_{n+1}$, and $H = K \cup (\bigcup_{t \in E_n} B_t)$ (H is a closed set by assumptions $(1)_k$, $k = 0, \ldots, n$). Then there exist $h \in C(X)$ and $\{B_s | s \in D'_{n+1}\}$, a collection of closed balls with each B_s centered at s, such that

(a) $\{B_s | s \in D'_{n+1}\}$ is a discrete collection of sets,

(b) $(\bigcup_{s \in D'_{n+1}} B_s) \cap (K \cup (\bigcup_{t \in E_n} B_t)) = \varnothing$,

(c) $h = f_n$ in $X - \bigcup_{s \in D'_{n+1}} B_s$,

(d) $f_n \leqslant h < f_n + \varepsilon/2^{n+3}$ everywhere in X,

(e) $h(x) < h(s)$ for every $s \in D'_{n+1}$ and $x \in B_s - \{s\}$.

Now, consider D''_{n+1}. If $D''_{n+1} = \varnothing$, then the collection $\{B_s | s \in D_{n+1}\}$ has already been defined; moreover, letting $f_{n+1} = h$, it is clear, by (a)–(e) and $(5)_n$, that conditions $(1)_{n+1}$–$(6)_{n+1}$ are fulfilled.

Hence, suppose that $D''_{n+1} \neq \varnothing$ and, according to Lemma 1, let $\{U_s | s \in D''_{n+1}\}$ be any discrete collection of open balls, with each U_s centered at s. Denote by Δ_s, for each $s \in D''_{n+1}$, the set $\{t \in E_n | s \in B_t\}$. By $(1)_k$, $k = 0, \dots, n$, Δ_s contains at most $n + 1$ elements. Let $\tau_s = \min\{f_n(t) - f_n(s) | t \in \Delta_s\}$. By $(5)_n$, τ_s is a positive number. Also denote by C_s the union of those balls B_t, $t \in E_n \cup D'_{n+1}$, which do not contain s. By $(1)_k$, $k = 0, \dots, n$, and (a) (if $D'_{n+1} \neq \varnothing$), C_s is a closed set. Then apply Lemma 2 with $\varphi = h$, $\eta = \eta_s = \min\{\varepsilon/2^{n+3}, \tau_s\}$, $D = \{s\}$, $H = K \cup C_s \cup \Delta_s \cup (X - U_s)$. Accordingly, there exist $h_s \in C(X)$ and B_s, a closed ball with center at s, such that

$(\beta)_s$ $B_s \subset U_s$, $B_s \cap (K \cup C_s \cup \Delta_s) = \varnothing$,

$(\gamma)_s$ $h_s = h$ in $X - B_s$,

$(\delta)_s$ $h \leqslant h_s < h + \eta_s$ everywhere in X,

$(\varepsilon)_s$ $h_s(x) < h_s(s)$ for every $x \in B_s - \{s\}$.

Do this for each $s \in D''_{n+1}$. Then the collection $\{B_s | s \in D_{n+1}\}$ has been defined. Moreover, having in mind condition $(\beta)_s$, $s \in D''_{n+1}$, the discreteness of $\{U_s | s \in D''_{n+1}\}$ and also, if $D'_{n+1} \neq \varnothing$, conditions (a)–(b), it is clear that $(1)_{n+1}$, $(2)_{n+1}$, and $(6)_{n+1}$ are fulfilled. Let f_{n+1}: $X \to \mathbf{R}$ be defined as follows:

$$f_{n+1}(x) = h(x) \quad \text{if } x \in X - \bigcup_{s \in D''_{n+1}} B_s,$$

$$f_{n+1}(x) = h_s(x) \quad \text{if } x \in B_s, s \in D''_{n+1}.$$

By $(1)_{n+1}$, f_{n+1} is well defined; also, by $(1)_{n+1}$ and $(\gamma)_s$, $s \in D''_{n+1}$, it is a continuous function. Moreover, using conditions $(\delta)_s$, $s \in D''_{n+1}$, and, as usual, (c)–(d), if $D'_{n+1} \neq \varnothing$, it is easy to check the validity of $(3)_{n+1}$–$(4)_{n+1}$. Let us show that also $(5)_{n+1}$ is satisfied. This will conclude the inductive argument. Let $s \in E_{n+1}$, $x \in B_s - \{s\}$. First assume $s \in E_n$, so $f_{n+1}(s) = f_n(s)$ by $(2)_{n+1}$–$(3)_{n+1}$. If $x \notin \bigcup_{t \in D_{n+1}} B_t$, then $f_{n+1}(x) = f_n(x) < f_n(s) = f_{n+1}(s)$ by $(3)_{n+1}$ and $(5)_n$. If $x \in B_t$ for some $t \in D_{n+1}$, then $t \in B_s$ by $(6)_{n+1}$; hence, $t \in D''_{n+1}$ and $s \in \Delta_t$; it follows that, by $(\varepsilon)_t$, $(\delta)_t$, and also (b)–(c), if $D'_{n+1} \neq \varnothing$, then

$$\begin{aligned} f_{n+1}(x) &= h_t(x) \leqslant h_t(t) < h(t) + \tau_t \\ &= f_n(t) + \tau_t \leqslant f_n(s) = f_{n+1}(s). \end{aligned}$$

Now consider the case $s \in D_{n+1}$. Then we again get $f_{n+1}(x) < f_{n+1}(s)$, using condition (e) if $s \in D'_{n+1}$ or condition $(\varepsilon)_s$ if $s \in D''_{n+1}$.

At this point we are in a position to define the function g that we are looking for. Let $g(x) = \lim_n f_n(x)$ for every $x \in X$. By conditions $(4)_n$, $n = 0, 1, 2, \dots$, g is a well-defined, continuous function satisfying (ii). Also, by $(2)_n$–$(3)_n$, $n = 0, 1, \dots$, it is clear that (i) is fulfilled. Let us check (iii). To this end it is enough to show that

$g(x) < g(s)$ for every $s \in S$ and $x \in B_s - \{s\}$. Let s and x be fixed as above. Suppose $s \in D_\nu$; then $f_\nu(s) = f_{\nu+1}(s) = \cdots = g(s)$ by $(2)_n$–$(3)_n$, $n = \nu + 1, \nu + 2, \ldots$. Moreover, let

$$L = \left\{ n \,\middle|\, x \in \bigcup_{t \in D_n} B_t \right\}$$

so $\nu \in L$. We distinguish two cases, according as $\nu = \max L$ or otherwise. In the first case, by $(3)_n$, $n = \nu + 1, \nu + 2, \ldots$, and $(5)_\nu$, we have $g(x) = f_\nu(x) < f_\nu(s) = g(s)$. In the second case, let $m^* \in L$ and $t \in D_{m^*}$ be fixed in such a way that $m^* > \nu$ and $x \in B_t$. Then, for every $m \geqslant m^*$, by $(5)_m$ and $(2)_n$–$(3)_n$, $n = m^* + 1, m^* + 2, \ldots$, we get $f_m(x) \leqslant f_m(t) = f_{m^*}(t)$. Consequently, by $(5)_{m^*}$, $g(x) \leqslant f_{m^*}(t) < f_{m^*}(s) = g(s)$. This concludes the proof. □

Let us point out that Theorem 1 implies that for every σ-discrete set $S \subset X$, the set $\mathcal{M}_S$ of all functions $f \in C(X)$, such that $M(f) \supset S$, is dense in $C(X)$ when the latter is endowed with the topology τ. In particular, we get the following

COROLLARY 1. *Let $\mathcal{M}$ be the set of all functions $f \in C(X)$ such that $M(f)$ is dense in X. Then $\mathcal{M}$ is a dense subset of $C(X)$ endowed with the topology τ.*

PROOF. This follows from the previous remark and the obvious fact that $\mathcal{M} \supset \mathcal{M}_S$ for every σ-discrete dense subset S of X. The existence of such a set S is deduced from the existence of a σ-discrete base for X. A more straightforward proof can be achieved as follows. For every $n = 1, 2, \ldots$ let D_n be any subset of X which is maximal with respect to the property: $x, y \in D_n$, $x \neq y \Rightarrow d(x, y) \geqslant 1/n$; then $S = \bigcup_{n=1}^{\infty} D_n$ works. □

We also have the following proposition which solves negatively Problem 3.2 of **[2]**.

COROLLARY 2. *Let the metric space X be perfect. Then $\mathcal{R}(X)$ is a dense subset of $C(X)$ endowed with the topology τ.*

PROOF. Since X is perfect, we have $\mathcal{M} \subset \mathcal{R}(X)$. Then the density of $\mathcal{R}(X)$ follows from Corollary 1. □

In view of the above corollary, we have that, for a metric space X, the perfectness of X and the fact that $\mathcal{R}(X) \neq \varnothing$ are equivalent.

Finally, we would like to show how Theorem 1 enables us to solve positively Problem 3.1 of **[2]**; namely, whether the assumption of the local connectedness of X can be dropped in Theorem A. As a matter of fact, Theorem 1 allows a further improvement of Theorem A—namely, obtaining the density of $\mathcal{R}(X, Y)$ in $C(X, Y)$ with respect to τ.

THEOREM 2. *Let X be perfect. Then for every $g \in C(X, Y)$ and every $\varepsilon \in C(X)$, with $\varepsilon > 0$ everywhere in X, there exists $g_\varepsilon \in \mathcal{R}(X, Y)$ such that $\|g(x) - g_\varepsilon(x)\| < \varepsilon(x)$ for each $x \in X$. Moreover, $g_\varepsilon(X) \subset \operatorname{conv}(g(X))$ provided that g is not constant.*

PROOF. As it is not restrictive, we may assume $0 \in g(X)$ and $\varepsilon \leqslant 1$ everywhere in X. Let Σ be any σ-discrete dense subset of X, and let Ω denote the set $\bigcup_{y \in Y} \operatorname{int} g^{-1}(y)$. Also, let $\eta \in C(X)$ be defined by $\eta(x) = \varepsilon(x)/(1 + \|g(x)\|)$, $x \in X$. By Theorem 1 there exists $\lambda \in C(X)$ such that (i) $\lambda = -1$ in $X - \Omega$, (ii) $-1 \leqslant \lambda < -1 + \eta$ everywhere in X, and (iii) $M(\lambda) \supset \Sigma \cap \Omega$. If g is a constant, i.e, $g = 0$ everywhere in X, we take $g_\varepsilon = (1 + \lambda)\bar{y}$, with $\bar{y}$ any fixed element of Y of norm one. If g is not constant we fix $\bar{y} \in \mathrm{conv}(f(X))$, with $0 < \|\bar{y}\| \leqslant 1$, and define g_ε as follows: $g_\varepsilon(x) = -\lambda(x)g(x)$ if $x \in X - \operatorname{int} g^{-1}(0)$, and $g_\varepsilon(x) = (1 + \lambda(x))\bar{y}$ if $x \in \operatorname{int} g^{-1}(0)$. Then it is easy to check that g_ε satisfies the thesis of the theorem. □

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