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A NOTE ON INFINITE-DIMENSIONAL SPACES DEFINED BY TOPOLOGICAL GAMES

YASUNAO HATTORI

ABSTRACT. We get the following result: For a totally normal space X, X is a strongly countable dimensional space if and only if X is an **F(dim)**-like space. This result answers a question of R. Telgársky.

Let **K** denote a class of spaces which are hereditary with respect to closed subspaces and $\sigma\mathbf{K}$ a class of all spaces which are countable unions of closed subspaces belonging to **K**. Recall from [T] that a space is called **K**-*like* if Player I has a winning strategy in $G(\mathbf{K}, X)$. In [T, Question 4.10], R. Telgársky asked whether X is **K**-like if it is $\sigma\mathbf{K}$-like. Under the assumption of CH, he gave a negative answer to this question (see [$\mathbf{Y_1}$, Example 5.1]). He used the Lusin set to construct the example. However K. Kunen [K] showed that MA + $\neg$CH implies that there are no (useful) Lusin sets. Hence the assumption of CH seems to be essential to the construction of the R. Telgársky's example. In this note, we answer the above question without any set-theoretic assumption beyond ZFC. Let **F(dim)** denote the class of all finite-dimensional spaces (in the sense of the covering dimension dim) and N the set of all natural numbers. For each n, i, $k \in N$ and $\alpha = \langle k_1,\dots,k_n\rangle \in N^n$, let

$$\alpha_{-i} = \langle k_1,\dots,k_{n-i}\rangle, \quad \alpha_{-n} = \varnothing \quad \text{and} \quad \alpha \oplus k = \langle k_1,\dots,k_n, k\rangle.$$

For each admissible sequence $T = \langle E_0, E_1,\dots,E_{2n}\rangle$ in $G(\mathbf{K}, X)$ and a closed set F of X with $F \cap s(T) = \varnothing$, let $\langle T; s(T); F\rangle$ denote an admissible sequence $\langle E_0, E_1,\dots,E_{2n}, s(T), F\rangle$ in $G(\mathbf{K}, X)$, where s is a strategy of Player I. We refer the readers to [T] for the descriptions and details of the topological games and to [P] for the dimension theory.

To answer the above question, we need the following result which is of interest in itself.

THEOREM. *Let X be a totally normal space. Then X is strongly countable dimensional (i.e. $X \in \sigma\mathbf{F(dim)}$) if and only if X is* **F(dim)**-*like.*

PROOF. Let X be an **F(dim)**-like space and s a winning strategy of Player I in $G(\mathbf{F(dim)}, X)$. Without loss of generality we can assume that X is infinite-dimensional. Let $T(\langle \varnothing, \varnothing\rangle) = \{\langle E_0\rangle\}$ and $Z(\langle \varnothing, \varnothing\rangle) = s(E_0)$, where $E_0 = X$. Since X

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is totally normal, for each $n \in N$ and $\langle \alpha, \beta \rangle \in N^n \times N^n$, we can inductively construct families $\mathscr{S}(T, \langle \alpha, \beta \rangle)$ of closed subsets of X for each $T \in \mathscr{T}(\langle \alpha_{-1}, \beta_{-1} \rangle)$, families $\mathscr{T}(\langle \alpha, \beta \rangle)$ of admissible sequences in the game $G(\mathbf{F(dim)}, X)$ and subsets $Z(\langle \alpha, \beta \rangle)$ of X which satisfy the following conditions:

(1) $\mathscr{S}(T, \langle \alpha, \beta \rangle)$ is locally finite at each point of $E_{2n-2} - s(T)$, where $T = \langle E_0, \ldots, E_{2n-2} \rangle$.

(2) $\bigcup \{ \mathscr{S}(T, \langle \alpha_{-1} \oplus k, \beta_{-1} \oplus m \rangle) \colon k, m \in N \}$ is a cover of $E_{2n-2} - s(T)$, where $T = \langle E_0, \ldots, E_{2n-2} \rangle$.

(3)

$$\mathscr{T}(\langle \alpha, \beta \rangle) = \begin{cases} \{ \langle T; s(T); F \rangle \colon T \in \mathscr{T}(\langle \alpha_{-1}, \beta_{-1} \rangle) \text{ and } F \in \mathscr{S}(T, \langle \alpha, \beta \rangle) \}, \\ \text{if } \mathscr{T}(\langle \alpha_{-1}, \beta_{-1} \rangle) \neq \varnothing \text{ and } \mathscr{S}(T, \langle \alpha, \beta \rangle) \neq \varnothing, \\ \varnothing, \quad \text{otherwise.} \end{cases}$$

(4) $\dim s(T) \leqslant m$ for each $T \in \mathscr{T}(\langle \alpha, \beta \rangle)$, whenever $\beta = \beta_{-1} \oplus m$.

(5) $$Z(\langle \alpha, \beta \rangle) = Z(\langle \alpha_{-1}, \beta_{-1} \rangle) \cup \bigcup \{ s(T) \colon T \in \mathscr{T}(\langle \alpha, \beta \rangle) \}.$$

Now, for each $\langle \alpha, \beta \rangle \in N^n \times N^n$ and $n \in N$, put

$$\mathscr{S}(\langle \alpha, \beta \rangle) = \bigcup \{ \mathscr{S}(T; \langle \alpha, \beta \rangle) \colon T \in \mathscr{T}(\langle \alpha_{-1}, \beta_{-1} \rangle) \}.$$

Claim 1. *$\mathscr{S}(\langle \alpha, \beta \rangle)$ is locally finite at each point of $X - Z(\langle \alpha_{-1}, \beta_{-1} \rangle)$.*

PROOF. Let $x \in X - Z(\langle \alpha_{-1}, \beta_{-1} \rangle)$. Without loss of generality we can assume that $\langle \alpha, \beta \rangle \in N^n \times N^n$ and $n \geqslant 2$. Since $x \notin Z(\langle \alpha_{-1}, \beta_{-1} \rangle) \supset Z(\langle \alpha_{-2}, \beta_{-2} \rangle)$, there is a neighbourhood U of x which meets at most finitely many members of $\mathscr{S}(\langle \alpha_{-1}, \beta_{-1} \rangle)$, by the inductive assumption. Let $F \in \mathscr{S}(T, \langle \alpha_{-1}, \beta_{-1} \rangle)$ with $F \cap U \neq \varnothing$, where $T = \langle E_0, \ldots, E_{2n-4} \rangle \in \mathscr{T}(\langle \alpha_{-2}, \beta_{-2} \rangle)$. By the condition (1), $\mathscr{S}(T', \langle \alpha_{-1}, \beta_{-1} \rangle)$ is locally finite in $F - s(T')$, where $T' = \langle T; s(T); F \rangle$ and hence so in $X - s(T')$. By the conditions (3) and (5), it follows that $\mathscr{S}(T', \langle \alpha_{-1}, \beta_{-1} \rangle)$ is locally finite at each point of $X - Z(\langle \alpha_{-1}, \beta_{-1} \rangle)$. Let $U(F)$ be a neighbourhood of x which meets at most finitely many members of $\mathscr{S}(T', \langle \alpha_{-1}, \beta_{-1} \rangle)$. Put

$$V = U \cap \left(\bigcap \{ U(F) \colon F \in \mathscr{S}(\langle \alpha_{-1}, \beta_{-1} \rangle) \text{ and } U \cap F \neq \varnothing \} \right).$$

It is clear that V is a neighbourhood of x which meets at most finitely many members of $\mathscr{S}(\langle \alpha, \beta \rangle)$.

Claim 2. *$Z(\langle \alpha, \beta \rangle)$ is a closed finite-dimensional subspace of X.*

PROOF. We shall prove by induction. It is obvious that $Z(\langle \varnothing, \varnothing \rangle)$ is a closed finite-dimensional subspace of X. Let $n \in N$ and $\langle \alpha, \beta \rangle \in N^n \times N^n$. If $\mathscr{T}(\langle \alpha, \beta \rangle) = \varnothing$, then $Z(\langle \alpha, \beta \rangle) = Z(\langle \alpha_{-1}, \beta_{-1} \rangle)$. Assume that $\mathscr{T}(\langle \alpha, \beta \rangle) \neq \varnothing$. From Claim 1, it is easy to see that $Z(\langle \alpha, \beta \rangle)$ is a closed subset of X. Now, let $\beta = \beta_{-1} \oplus m$ and F be a closed subset of $Z(\langle \alpha, \beta \rangle)$ with $F \cap Z(\langle \alpha_{-1}, \beta_{-1} \rangle) = \varnothing$. From Claim 1 and the condition (5), $\{ s(T) \cap F \colon T \in \mathscr{T}(\langle \alpha, \beta \rangle) \}$ is a locally finite closed cover of F. For each $T \in \mathscr{T}(\langle \alpha, \beta \rangle)$, $\dim s(T) \leqslant m$ by the condition (4). Thus $\dim s(T) \cap F \leqslant m$

and hence $\dim F \leqslant m$. Hence $\dim Z(\langle \alpha, \beta \rangle) \leqslant \max\{\dim Z(\langle \alpha_{-1}, \beta_{-1} \rangle),\ m\} < \infty$ by the inductive assumption. Put

$$Z = Z(\langle \varnothing, \varnothing \rangle) \cup \left(\bigcup \{ Z(\langle \alpha, \beta \rangle) : \langle \alpha, \beta \rangle \in N^n \times N^n, n \in N \} \right).$$

To complete the proof of the "if" part it is enough to verify that $X = Z$. Assume that $x_0 \in X - Z$. Since $x_0 \notin Z(\langle \varnothing, \varnothing \rangle) = s(E_0)$, there are $k_1, m_1 \in N$ and $E_2 \in \mathscr{S}(\langle E_0 \rangle, \langle \langle k_1 \rangle, \langle m_1 \rangle \rangle)$ such that $x_0 \in E_2$ by the condition (2). Put $E_1 = s(E_0)$. Since $\langle E_0, E_1, E_2 \rangle \in \mathscr{T}(\langle k_1 \rangle, \langle m_1 \rangle)$, it follows that $E_3 \subset Z(\langle k_1 \rangle, \langle m_1 \rangle)$, where $E_3 = s(E_0, E_1, E_2)$. Hence $x_0 \in E_2 - E_3$. Continuing in this manner we get a play $\langle E_0, E_1, \ldots \rangle$ of $G(\mathbf{F(dim)}, X)$ such that $E_1 = s(E_0)$, $E_{2n+1} = s(E_0, E_1, \ldots E_{2n})$ for each $n \in N$ and $x_0 \in \bigcap \{ E_{2n} : n \in N \}$. This is a contradiction. The "only if" part is obvious. The proof is complete.

The following example answers [**T**, Question 4.10] negatively.

EXAMPLE. *There is a hereditarily paracompact* σ**F(dim)**-*like space which is not* **F(dim)**-*like.*

PROOF. Let K_ω denote the subspace of the Hilbert cube I^ω consisting of all points in I^ω which have only finitely many nonzero coordinates. Let X be a space $(I^\omega)_{K_\omega}$ defined as the set I^ω with the new topology generated by the base consisting of all sets of the form $U \cup M$, where U is an open set of the Hilbert cube I^ω and $M \subset I^\omega - K_\omega$. Since K_ω is a strongly countable dimensional closed subspace of X and $\dim X - K_\omega = 0$, X is a hereditarily paracompact σ**F(dim)**-like space. But X is not a strongly countable dimensional space (see [**EP**, Example 3.4]). Hence X is not **F(dim)**-like by the above theorem. This completes the proof.

REMARKS. 1. We can replace the assumption "totally normal" by "normal and hereditarily subparacompact" in the above theorem. This proof is quite parallel to the above one.

2. Recently Y. Yajima gave a positive partial answer to the above question [$\mathbf{Y_2}$, Theorem 2.1]: Let X be a P-space (in the sense of Morita [**M**]). Then X is **K**-like if X is σ**K**-like.

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DEPARTMENT OF MATHEMATICS, OSAKA KYOIKU UNIVERSITY, TENNOJI, OSAKA, 543, JAPAN