TRANSLATION PROPERTIES OF SETS OF POSITIVE UPPER DENSITY

VITALY BERGELSON AND BENJAMIN WEISS

Abstract. Generalizing a result of Raimi we show that there exists a set \( E \subset \mathbb{N} \) such that if \( A \subset \mathbb{N} \) is a set with positive upper density, then there exists a number \( k \in \mathbb{N} \) such that \( d^*(A + k \cap E) > 0 \) and \( d^*(A + k \cap E^c) > 0 \). Some extensions and further results are also obtained.

The purpose of this note is to generalize the following theorem due to Raimi (see [1]).

**Theorem.** There exists a set \( E \subset \mathbb{N} \) such that, whenever \( r \in \mathbb{N} \) and \( \mathbb{N} = \bigcup_{1 \leq i \leq r} D_i \), there exists \( 1 \leq i \leq r \) and \( k \in \mathbb{N} \) with
\[
| (D_i + k) \cap E | = \omega \quad \text{and} \quad | (D_i + k) \cap E^c | = \omega.
\]

Raimi's proof used a topological result about \( \mathbb{N} \). Another proof was given by Ryll-Nardzewski [2]. See also [3].

Raimi's theorem is topological in nature and it is natural to ask whether a density version holds.

The upper density \( d^*(A) \) of a set \( A \subset \mathbb{N} \) is defined by
\[
d^*(A) := \limsup_{n \to \infty} \frac{|A \cap [1,n]|}{n},
\]
where \( [1,n] = \{1,\ldots,n\} \). If the limit exists and is positive, then we say that \( A \) has positive density \( d(A) > 0 \). If \( \mathbb{N} = \bigcup_{1 \leq i \leq r} D_i \) then at least one of the sets \( D_i \) has positive upper density. Thus the following theorem is clearly a strengthening of Raimi's result.

**Theorem 1.** There exists a set \( E \subset \mathbb{N} \) such that for any \( A \subset \mathbb{N} \) with \( 0 < d^*(A) \) there exists a \( k \in \mathbb{N} \) such that
\[
d^*(A + k \cap E) > 0 \quad \text{and} \quad d^*(A + k \cap E^c) > 0.
\]
In fact, the assertion of Theorem 1 holds for every normal set \( E \subset \mathbb{N} \) (see definition below). Theorem 1 is actually a corollary of the following

**THEOREM 1'**. If \( E \subset \mathbb{N} \) is normal and \( A \subset \mathbb{N} \) has positive upper density \( d^*(A) \) then

\[
(*) \quad d^*\left( (A + k) \cap E \right) > 0 \quad \text{and} \quad d^*\left( (A + k) \cap E^c \right) > 0
\]

holds for all \( k \in \mathbb{Z} \) with at most \([-\log_2 d^*(A)]\) exceptions.

An even stronger result holds, namely

**THEOREM 2**. If \( E \subset \mathbb{N} \) is normal, \( A \subset \mathbb{N} \) has upper density \( d^*(A) \) and \( \epsilon > 0 \), then

\[
d^*\left( (A + k) \cap E \right) > \frac{1}{2}d^*(A) - \epsilon \quad \text{and} \quad d^*\left( (A + k) \cap E^c \right) > \frac{1}{2}d^*(A) - \epsilon
\]

holds for all \( k \in \mathbb{Z} \) with at most \( [d^*(A)/4\epsilon^2] \) exceptions.

Before presenting the proofs of Theorems 1 and 2, we give the definition and some basic properties of normal sets.

To any set \( A \subset \mathbb{N} \) we attach the \((0,1)\)-sequence \( a_n = 1_A(n) \) which is its indicator function.

**DEFINITION.** Let \( \{a_n\}_{n=1}^\infty \) be a \((0,1)\)-sequence. Let \( B_k = b_1b_2 \cdots b_k, \ k \geq 1, \) be any \((0,1)\)-word of length \( k \). Denote by \( D(B_k, m) \) the number of occurrences of the block \( B_k \) as a sub-block in the block \( a_1a_2 \cdots a_m \), i.e.

\[
D(B_k, m) = \left| \{ n \in \{1, \ldots, m-k+1\} : a_{n+j-1} = b_j \text{ for } 1 \leq j \leq k \} \right|,
\]

the sequence \( \{a_n\}_{n=1}^\infty \) is normal if \( \lim_{m \to \infty} D(B_k, m)/m = 2^{-k} \) for all \( k \geq 1 \) and all \( B_k \).

A set \( A \subset \mathbb{N} \) is normal if \( 1_A(n) \) is a normal sequence.

It is, perhaps, not obvious that such sets do exist, but actually almost every \((0,1)\)-sequence is normal (if one views \((0,1)\)-sequences as dyadic expansions of numbers in \([0, 1]\) with usual Lebesgue measure). There are also numerous explicit constructions of normal sequences (see \([4-6]\)).

For example

\[
1 \quad 10 \quad 11 \quad 100 \quad 101 \quad 110 \cdots
\]

is a normal sequence (this sequence is formed by the sequence 1, 2, 3, \ldots written in base 2).

If \( E \) is a normal set, then obviously \( d(E) = d(E^c) = \frac{1}{2} \).

If \( E \) is a normal set, then \( d(E \cap (E + k)) = \frac{1}{4} \) for all \( k \in \mathbb{Z} \setminus \{0\} \). To see this, note that \( 1_{E \cap (E + k)}(n) = 1 \) iff \( n \in E \) and \( n - k \in E \). Each \((0,1)\)-word \( i_1i_2 \cdots i_{k+1} \) of length \( k + 1 \) appears in \( E \) with frequency \( 1/2^{k+1} \) and, in exactly \( 2^{k-1} \) of these words, \( i_1 = i_{k+1} = 1 \). So, the frequency of those \( n \) that satisfy \( n \in E \) and \( n - k \in E \) is equal to \( 2^{k-1}/2^{k+1} = \frac{1}{4} \).

In the same fashion one shows that if \( E \) is normal set, then

\[
d\left( E \cap (E + k_1) \cap (E + k_2) \cap \cdots \cap (E + k_m) \right) = 2^{-(m+1)}
\]

for any integers \( 0 < k_1 < k_2 < \cdots < k_m \).
It is not difficult to see that the same holds if we replace some of the sets $E + k_i$ by $E^c + k_i$. So we have the following

**Lemma 1.** Let $E$ be a normal set and let

$$E^\alpha = \begin{cases} E & \text{if } \alpha = 1, \\ E^c & \text{if } \alpha = -1. \end{cases}$$

Then for any distinct integers $k_1, k_2, \ldots, k_m$ and any $(-1, 1)$-word $\alpha_1 \alpha_2 \cdots \alpha_m$,

$$d \left( \bigcap_{i=1}^m (E^{\alpha_i} + k_i) \right) = 2^{-m}.$$  

**Proof of Theorem 1.** Let $k_1, \ldots, k_m$ be distinct integers for which (*) fails. That is, for each $1 \leq i \leq m$ there is an $\alpha_i \in \{-1, 1\}$ such that $d^*((A + k_i) \cap E^{\alpha_i}) = 0$. Shifting both $A + k_i$ and $E^{\alpha_i} + k_i$ units to the left, we obtain

$$d^* (A \cap (E^{\alpha_i} - k_i)) = 0, \quad i = 1, 2, \ldots, m,$$

and therefore

$$d^* \left( A \cap \bigcup_{i=1}^m (E^{\alpha_i} - k_i) \right) = 0.$$

This, in turn, implies

$$d^*(A) = d^* \left( A \cap \left( X \setminus \bigcup_{i=1}^m (E^{\alpha_i} - k_i) \right) \right)$$

$$= d^* \left( A \cap \bigcap_{i=1}^m (E^{-\alpha_i} - k_i) \right)$$

$$\leq d^* \left( \bigcap_{i=1}^m (E^{-\alpha_i} - k_i) \right) = 2^{-m}$$

(see Lemma 1), and therefore $-\log_2 d^*(A) \geq m$.

**Lemma 2.** Let $(X, B, \lambda)$ be a probability space, and let $\mathcal{E}$ be a (finite or infinite) collection of measurable subsets of $X$, such that, for some $\delta > 0, |\lambda(E) - \frac{1}{2}| \leq \delta$ for all $E \in \mathcal{E}$ and $|\lambda(E \cap F) - \frac{1}{4}| \leq \delta$ for any two distinct set $E, F \in \mathcal{E}$. If $A \subset X$ is measurable and $\epsilon > \sqrt{2 \delta \lambda (A)}$, then the inequality

$$|\lambda (A \cap E) - \frac{1}{2} \lambda (A)| < \epsilon \quad \text{(or, equivalently } |\lambda (A \cap E^c) - \frac{1}{2} \lambda (A)| < \epsilon)$$

holds for $E \in \mathcal{E}$ with at most $\lambda (A)/2 (\epsilon^2 - 2 \delta \lambda (A))$ exceptions.

**Proof of Lemma 2.** First note that if $E, F \in \mathcal{E}, E \neq F$, then

$$|\lambda (E^c) - \frac{1}{2}| \leq \delta, \quad |\lambda (E \cap F^c) - \frac{1}{4}| \leq 2 \delta,$$

$$|\lambda (E^c \cap F) - \frac{1}{4}| \leq 2 \delta, \quad |\lambda (E \cap F^c) - \frac{1}{4}| \leq 3 \delta,$$

and therefore

$$|\lambda (E \cap F) - \lambda (E^c \cap F) - \lambda (E \cap F^c) + \lambda (E^c \cap F^c)| \leq 8 \delta.$$
Using characteristic functions, we can rewrite the last inequality as
\[ \left| \int (2 \mathbb{1}_E - 1)(2 \mathbb{1}_F - 1) \, d\lambda \right| = \left| \int (1_E - 1_E')(1_F - 1_{E'}) \, d\lambda \right| \]
\[ = \left| \int (1_E 1_{E'} - 1_E 1_{F'} - 1_E 1_{E'} + 1_{E'} 1_{F'}) \, d\lambda \right| \]
\[ = \left| \int (1_{E \cap F} - 1_{E' \cap F} - 1_{E \cap F'} + 1_{E' \cap F'}) \, d\lambda \right| \leq 8\delta. \]

Define
\[ \mathcal{E}_+ = \{ E \in \mathcal{E} : \lambda(A \cap E) \geq \frac{1}{2}\lambda(A) + \varepsilon \}, \]
\[ \mathcal{E}_- = \{ E \in \mathcal{E} : \lambda(A \cap E) \leq \frac{1}{2}\lambda(A) - \varepsilon \}. \]

Lemma 2 asserts that \( |\mathcal{E}_+ \cup \mathcal{E}_-| \leq \lambda(A)/2(\varepsilon^2 - 2\delta\lambda(A)) \). We shall actually show that
\[ \max(|\mathcal{E}_+|, |\mathcal{E}_-|) \leq \lambda(A)/4(\varepsilon^2 - 2\delta\lambda(A)). \]

We shall carry out the calculations for \( \mathcal{E}_+ \) only. Suppose \( E_1, \ldots, E_n \) are distinct sets in \( \mathcal{E}_+ \). We denote by \( \mathbb{1}_i \) the characteristic function of \( E_i, 1 \leq i \leq n \).

From the definition of \( \mathcal{E}_+ \) we obtain
\[ \varepsilon \leq \frac{1}{n} \sum_{i=1}^{n} \lambda(A \cap E_i) - \frac{1}{2}\lambda(A) \]
\[ = \int \left( \frac{1}{n} \sum_{i=1}^{n} 1_{A \cap E_i} - \frac{1}{2} 1_A \right) \, d\lambda \]
\[ = \int 1_A \left( \frac{1}{n} \sum_{i=1}^{n} 1_{E_i} - \frac{1}{2} \right) \, d\lambda \]
\[ = \int 1_A \left( \frac{1}{2n} \sum_{i=1}^{n} (2 \mathbb{1}_i - 1) \right) \, d\lambda. \]

Applying the classical Cauchy-Schwarz inequality we can continue:
\[ \leq \left[ \int 1_A^2 \, d\lambda \cdot \int \frac{1}{4n} \left( \sum_{i=1}^{n} (2 \mathbb{1}_i - 1)^2 \right) \, d\lambda \right]^{1/2} \]
\[ = \left[ \int 1_A \, d\lambda \cdot \frac{1}{4n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \int (2 \mathbb{1}_i - 1)(2 \mathbb{1}_j - 1) \, d\lambda \right]^{1/2} \]
\[ = \left[ \lambda(A) \cdot \frac{1}{4n^2} \left( \sum_{i=1}^{n} \int (2 \mathbb{1}_i - 1)^2 \, d\lambda + 2 \sum_{1 \leq i < j \leq n} \int (2 \mathbb{1}_i - 1)(2 \mathbb{1}_j - 1) \, d\lambda \right) \right]^{1/2}. \]

Observing that \( (2 \mathbb{1}_i - 1)^2 \equiv 1 \) and using inequality (1) we can continue:
\[ \leq \left[ \lambda(A) \cdot \frac{1}{4n^2} (n + n(n-1)8\delta) \right]^{1/2} \leq \left[ \lambda(A) \cdot \left( \frac{1}{4n^2} + 2\delta \right) \right]^{1/2}. \]
It follows that $\varepsilon^2 \leq \lambda(A)(1/4n + 2\delta)$, and therefore after elementary calculations we obtain $n \leq \lambda(A)/4(\varepsilon^2 - 2\delta\lambda(A))$.

The proof of the inequality for $\delta^e$ is essentially the same.

**Proof of Theorem 2.** Suppose $E \subseteq \mathbb{N}$ is a normal set and let $A \subseteq \mathbb{N}$ and $\varepsilon > 0$ be given.

For a set $B \subseteq \mathbb{N}$ we shall write $d_n(B) = \frac{1}{n}|B \cap [1, n]|$, so that $d^*(B) = \limsup_{n \to \infty} d_n(B)$. Define

$$\mathcal{X} = \{ k \in \mathbb{Z} : d^*((A + k) \cap E) < \frac{1}{2}d^*(A) - \varepsilon \} = \{ k \in \mathbb{Z} : d^*(A \cap (E - k)) < \frac{1}{2}d^*(A) - \varepsilon \},$$

$$\mathcal{X}' = \{ k \in \mathbb{Z} : d^*((A + k) \cap E^c) \leq \frac{1}{2}d^*(A) - \varepsilon \} = \{ k \in \mathbb{Z} : d^*(A \cap (E^c - k)) \leq \frac{1}{2}d^*(A) - \varepsilon \}.$$

We shall prove Theorem 2 by showing that $\max(|\mathcal{X}|, |\mathcal{X}'|) \leq d^*(A)/4\varepsilon^2$. Suppose that $|\mathcal{X}| > d^*(A)/4\varepsilon^2$. Let $k_1, k_2, \ldots, k_n$ be distinct numbers in $\mathcal{X}$, $n > d^*(A)/4\varepsilon^2$. Choose a positive number $\delta$, such that $n > d^*(A)/4(\varepsilon^2 - 2\delta d^*(A))$ and let $\{m_i\}_{i=1}^\infty$ be an increasing sequence of positive integers such that $d^*(A) = \lim_{i \to \infty} d_{m_i}(A)$. Choose a number $i_0$ such that for all $i \geq i_0$ the following inequalities hold:

$$|d_{m_i}(N \cap (E - k_p)) - \frac{1}{2}| < \delta$$

for all $1 \leq p \leq n$,

$$|d_{m_i}(N \cap (E - k_p) \cap (E - k_q)) - \frac{1}{2}| < \delta$$

for all $1 \leq p < q \leq n$,

$$n > d_{m_i}(A)/4(\varepsilon^2 - 2\delta \cdot d_{m_i}(A)).$$

(The existence of a number $j_0$ is an immediate consequence of the normality of $E$.)

For $i \geq i_0$ and $1 \leq p \leq n$ put $A^i = A \cap [1, m_i]$; $E^i_p = (E - k_p) \cap [1, m_i]$. Note that $d_{m_i}$ is a probability measure on the set of all subsets of $[1, m_i]$.

Inequalities (2) can be rewritten as

$$|d_{m_i}(E^i_p) - \frac{1}{2}| < \delta$$

for all $1 \leq p \leq n$,

$$|d_{m_i}(E^i_p \cap E^i_q) - \frac{1}{2}| < \delta$$

for all $1 \leq p < q \leq n$,

$$n > d_{m_i}(A^i)/4(\varepsilon^2 - 2\delta \cdot d_{m_i}(A^i)).$$

By the proof of Lemma 2 there is at least one index $p_i$, $1 \leq p_i \leq n$, such that

$$d_{m_i}(A^i \cap E^i_{p_i}) > \frac{1}{2}d_{m_i}(A) - \varepsilon,$$

Since $p_i \in [1, n]$ for all $i \geq i_0$, there is an infinite set $I$ of indices and a number $p$ such that $p_i = p$ for all $i \in I$.

Passing to the upper limit as $i \to \infty$, $i \in I$, we obtain from (3) $d^*(A \cap E - p) > \frac{1}{2}d^*(A) - \varepsilon$ for some $p \in \mathcal{X}$ which contradicts the definition of $\mathcal{X}$. This shows that $|\mathcal{X}| \leq d^*(A)/4\varepsilon^2$. The proof that $|\mathcal{X}'| \leq d^*(A)/4\varepsilon^2$ is essentially the same and is left to the reader.

It is natural to ask whether or not the results obtained here can be generalized to other groups. As a matter of fact even the group structure is irrelevant, and one can...
establish the following result: Let $\Phi$ denote some countably infinite family of one-to-one mappings $\varphi: \mathbb{N} \to \mathbb{N}$ (not necessarily onto), that acts freely on $\mathbb{N}$, i.e. for $\varphi \neq \varphi'$ and all $n \in \mathbb{N}$, $\varphi(n) \neq \varphi'(n)$. Then there exists a set $E$ such that for all sets $A$ with positive upper density, both $d^*(A \cap \varphi^{-1}(E)) > 0$ and $d^*(A \cap \varphi^{-1}(E^c)) > 0$ hold for $\varphi \in \Phi$ with at most $[-\log_2 d^*(A)]$ exceptions. The proof goes along the lines of the proof of Theorem 1', to be sure the notion of normality of $E$ is defined now with respect to $\Phi$. Because of the independence of the underlying random variables, the fact that $\Phi$ has no structure presents no obstacle and one easily establishes the existence of $\Phi$-normal sets $E$ that satisfy the property:

For every finite set $\Phi_0 \subset \Phi$, and every choice of $a(\varphi) \in \{-1,1\}$, $\varphi \in \Phi_0$, we have

$$\lim_{n \to \infty} \frac{1}{n} \left| \{ i \leq n : \varphi(i) \in E^{a(\varphi)}, \varphi \in \Phi_0 \} \right| = 2^{-|\Phi_0|}$$

where as usual $E^1 = E$ and $E^{-1} = E^c$.

Then for $E$ one can take any $\Phi$ normal set. The details are straightforward and can safely be left to the reader.

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Institute of Mathematics and Computer Science, Hebrew University of Jerusalem, Givat Ram, 91904 Jerusalem, Israel

Current address (Vitaly Bergelson): Department of Mathematics, Ohio State University, Columbus, Ohio 43210

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