EXISTENCE OF SOLUTIONS IN A CONE
FOR NONLINEAR ALTERNATIVE PROBLEMS

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Abstract. Using the alternative method we present sufficient conditions for the existence of positive solutions to nonlinear equations at resonance and extend a well-known result of Cesari and Kannan.

Introduction. Cesari and Kannan [2] proved an abstract result in terms of the alternative method. Their result and some of its ramifications (see [1]) have been applied to a large class of problems at resonance to prove the existence of solutions.

Let $E$ be a Banach space. We say that $C$ is a cone in $E$ if $C$ is a nonempty, convex subset of $E$ such that $\lambda C \subset C$ for every $\lambda \geq 0$.

Here we prove the existence of solutions in a cone for equations at resonance of the form $Lu = Nu$, where $L$ is a linear operator and $N$ is a (nonlinear) operator. In the case when the cone is $E$, we obtain the well-known result of Cesari and Kannan [2].

In applications, for instance, if $L$ is an elliptic operator on a bounded domain $\Omega$ of $\mathbb{R}^n$, one usually takes $E$ as a subspace of $L^2(\Omega)$ and the cone $C = \{u \in E : u \geq 0 \text{ a.e. in } \Omega \}$.

Also, our result is related to that of Gaines and Santanilla [3] concerning the existence of solutions in a convex set.

Main result. Let $E$ and $F$ be Banach spaces with norms $\| \cdot \|_E$ and $\| \cdot \|_F$, respectively. Let $L : D(L) \subset E \rightarrow F$ be a linear operator and $N : E \rightarrow F$ a continuous (nonlinear) operator such that $N$ maps bounded sets into bounded sets. Assume that $C$ is a cone in $E$ and

$$\text{(1) there exists a continuous map } \gamma : E \rightarrow C \text{ such that } \gamma(c) = c \text{ for every } c \in C, \text{ and } \gamma \text{ maps bounded sets in } E \text{ into bounded sets in } E.$$ 

In addition, suppose that $L$ is a Fredholm map of index 0 and there exist projections $P : E \rightarrow E$, $Q : F \rightarrow F$, and a linear map $H : (I - Q)F \rightarrow (I - P)E$
satisfying

\[ H(I - Q)Lu = (I - P)u \quad \text{for every } u \in D(L), \]
\[ QLu = LPu \quad \text{for every } u \in D(L), \]
\[ LH(I - Q)Nu = (I - Q)Nu \quad \text{for every } u \in E. \]

Thus, it is well known that \( Lu = Nu \) is equivalent to the coupled system of equations

\[ QNu = 0 \quad \text{(bifurcation equation),} \]
\[ u = Pu + H(I - Q)Nu \quad \text{(auxiliary equation).} \]

We can write the spaces \( E \) and \( F \) as the direct sums \( E = E_0 \oplus E_1, F = F_0 \oplus F_1 \), where \( E_0 = PE, E_1 = (I - P)E, F_0 = QF, \) and \( F_1 = (I - Q)F \). Also, we assume

(3) \( E_0 = \text{Ker} L, F_1 = \text{Im} L, D(H) = \text{Im} L \) and \( \text{Im} H = E_1 \cap D(L) \).

(4) \( \dim E_0 = \dim F_0 < +\infty \).

(5) \( L \) is completely continuous.

(6) There exist continuous maps \( B: E \times F \to \mathbb{R} \) and \( J: F_0 \to E_0 \) such that \( B \) is bilinear, \( J \) is one-to-one and onto, and

(i) for \( v_0 \in F_0, v_0 = 0 \) iff \( B(u_0, v_0) = 0 \) for all \( u_0 \in E_0 \),

(ii) \( B(Jv_0, v_0) \geq 0 \) for every \( v_0 \in F_0 \) and \( B(Jv_0, v_0) = 0 \) iff \( v_0 = 0 \),

(iii) \( Jv_0 = 0 \) iff \( v_0 = 0 \),

(iv) \( B(u_0, J^{-1}v_0) = 0 \) iff \( u_0 = 0 \),

(v) \( B(u_0, v_0) = B(Jv_0, J^{-1}u_0) \) for every \( u_0 \in E_0, v_0 \in F_0 \).

**Remark.** If \( E \subset F \) and \( F \) is a Hilbert space with inner product \( \langle u, v \rangle \), then one can define \( B(u_0, v_0) = \langle u_0, v_0 \rangle \). Thus, if \( F = L^2(\Omega) \),

\[ B(u_0, v_0) = \int_\Omega u_0(x) \cdot v_0(x) \, dx. \]

For \( u \in E \) we write \( u = u_0 + u_1 \), with \( u_0 \in E_0, u_1 \in E_1 \). With this, the auxiliary and bifurcation equations become \( QN(u_0 + u_1) = 0 \) and \( u_1 = H(I - Q)N(u_0 + u_1) \), respectively. We are now in a position to prove our result.

**Theorem.** Let conditions (1)-(6) hold. In addition, assume there exists

(7) \( J_0 > 0 \) such that \( \|Nu\| \leq J_0 \) for every \( u \in C \),

(8) \( R_0 > 0 \) such that \( B(u_0, QN(u)) \leq 0 \) for every \( u = u_0 + u_1 \in C, \) with \( \|u_0\| = R_0 \) and \( u_1 = H(I - Q)N(u_0 + u_1) \), and

(9) \( r_0 > \|H(I - Q)\| \cdot J_0 \) such that \( (P + JQN)\gamma u \in C \) and \( H(I - Q)N\gamma(u) \in C \) for every \( u \in S \), where

\[ S = \{ u = u_0 + u_1 \in E: \|u_0\| \leq R_0, \|u_1\| \leq r_0 \}. \]

Then \( Lu = Nu \) has at least one solution \( u \in S \cap C \).

**Proof.** The set \( S \) is closed, bounded, and convex. Define the homotopy \( T: [0, 1] \times S \to E \) by

\[ T(\lambda, u) = \lambda P\gamma(u) + H(I - Q)N\gamma(u) + \lambda JQN\gamma(u). \]

Note that \( T(\lambda, \cdot) \) is compact for every \( \lambda \in [0, 1] \) since \( P \) and \( Q \) are projections with finite-dimensional range and \( H \) is compact. For \( \lambda = 0, T(0, u) = H(I - Q)N\gamma(u) \in E_1 \).
Thus, by (9),
\[ \|T(0, u)\| \leq \|H(I - Q)\| \cdot \|N\gamma(u)\| < r_0, \]
which shows that \( T(0, \delta S) \subseteq S \).

We shall now prove that \( T(\lambda, u) \neq u \) for every \((\lambda, u) \in [0, 1) \times \delta S\). Indeed, let \( T(\lambda, u) = u \) and, consequently,
\begin{align*}
(10) & \quad u_0 = \lambda P\gamma(u) + \lambda JQN\gamma(u), \\
(11) & \quad u_1 = H(I - Q)N\gamma(u).
\end{align*}
If \( u \in \delta S \), then either \( \|u_0\| = r_0 \) or \( \|u_0\| = R_0 \). In the first case, using (11), we get
\[ r_0 = \|u_1\| = \|H(I - Q)N\gamma(u)\| < r_0, \]
which is a contradiction.

In the second case, \( \|u_0\| = R_0 \). Hence, by (9), \((P + JQN)\gamma(u) \in C\) and \( u_0 \in C \) since \( C \) is a cone. Also by (9), \( u_1 = H(I - Q)N\gamma(u) \in C\) and, consequently, \( u = u_0 + u_1 \in C \). This implies, by the property of \( \gamma \), that \( \gamma(u) = u \) and \( u_0 = \lambda Pu + \lambda JQN\gamma(u) \). This last inequality is equivalent to \( (1 - \lambda)u_0 = \lambda JQN\gamma(u) \). We assume that \( \lambda > 0 \) since \( \lambda = 0 \) implies \((1 - \lambda)u_0 = 0\) and \( u_0 = 0 \). Thus, by (8) and (11), \( B(u_0, Q\gamma(u)) \leq 0 \). On the other hand,
\[ \lambda B(u_0, QNu) = \lambda B(JQN\gamma(u), J^{-1}u_0) = B((1 - \lambda)u_0, J^{-1}u_0) > 0, \]
which is again a contradiction. Therefore, \( T(\lambda, u) \neq u \) for every \((\lambda, u) \in [0, 1) \times \delta S\), and we can conclude [4, Theorem 4.4.11] that \( T(1, \cdot) \) has a fixed point. Hence, there exists \( u \in S \) satisfying
\[ u = P\gamma(u) + H(I - Q)N\gamma(u) + JQN\gamma(u). \]
Reasoning as before, \( u \in C \) and satisfies the auxiliary and bifurcation equations, that is, \( u \) is a solution of \( Lu = Nu \) such that \( u \in S \cap C \). This completes the proof of the Theorem.

If \( C = E \) we obtain the result of [2].

**COROLLARY.** Let conditions (1)–(6) hold. In addition, assume there exists
\begin{align*}
(7)' & \quad J_0 > 0 \text{ such that } \|Nu\| \leq J_0 \text{ for every } u \in E, \\
(8)' & \quad R_0 > 0 \text{ such that } B(u_0, QNu) \leq 0 \text{ for every } u = u_0 + u_1 \in E, \text{ with } \|u_0\| = R_0.
\end{align*}
and \( u_1 = H(I - Q)N(u_0 + u_1) \).

Then \( Lu = Nu \) has at least one solution.

For some particular cases of our result and applications to nonlinear boundary value problems, see [5].

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**References**


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