ALMOST EUCLIDEAN QUOTIENT SPACES OF SUBSPACES 
OF A FINITE-DIMENSIONAL NORMED SPACE 

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ABSTRACT. The main result of this article is Theorem 1 which states that a quotient 
space Y, dim Y = k, of a subspace of any finite dimensional normed space X, 
dim X = n, may be chosen to be d-isomorphic to a euclidean space even for 
k = \lfloor \lambda n \rfloor \; \text{for any fixed} \; \lambda < 1 \; \text{and} \; d \text{depending on} \; \lambda \; \text{only.}

The following theorem is proved.

1. Theorem. For every d > 1 there exists \( \lambda(d) > 0 \) such that every n-dimensional 
normed space X contains a k-dimensional quotient space F of a subspace E \subset X which satisfies

(i) \( d(F, l_2^k) < d, \)

(ii) \( \dim F = k \geq \lambda(d)n. \)

(Here \( d(F, l_2^k) \) denotes a Banach-Mazur distance between two normed spaces; i.e.,
\[
d(X, Y) = \inf\{ \|T\| \cdot \|T^{-1}\| \text{ over all linear isomorphisms } T: X \to Y \}.\]

Moreover, \( \lambda(d) \to 1 \text{ if } d \to \infty \) and, for large \( d, \lambda(d) = 1 - 3\sqrt{6}/\ln \ln d. \)

Remark 1. It is enough to prove Theorem 1 for large \( d \) only, because, as proved in 
[M1], any d-isomorphic copy of \( l_2^k \) contains, for any \( \epsilon > 0 \), a \((1 + \epsilon)\)-isomorphic copy of \( l_2^k \), where \( k \geq \kappa(\epsilon)m/d^2 \) and \( \kappa(\epsilon) > 0 \) depends on \( \epsilon > 0 \) only.

Remark 2. Of course, the theorem states that the dual \( E^* \subset E \subset X \) contains a 
subspace \( F^* \subset E^* \) which satisfies (i) and (ii) of the theorem.

Remark 3. In [M2] the theorem was proved with a logarithmic factor, and this 
theorem was formulated as a problem. We refer the reader to this paper for relevant 
discussion.

2. Notations. Let \( X \) be an n-dimensional normed space, i.e., \( R^n \) with the norm \( \| \cdot \| \), 
and let \( (x, y) \) be an inner product on \( X \); consequently, \( |x| = (x, x)^{1/2} \) is a euclidean 
norm on \( X \). For any \( x \in X \) let \((1/a)|x| \leq \|x\| \leq b|x|\) and \( M_r = \int_{x \in S^{n-1}} \|x\| d\mu(x), \)
where \( S^{n-1} = \{ x \in X : |x| = 1 \} \) and \( \mu(x) = \mu_{n-1}(x) \) is the normalized invariant 
(Haar) measure on \( S^{n-1} \). Let \( \|x\||^* = \sup_{y \neq 0}(|(x, y)|/\|y\|) \). Then \((1/b)|x| \leq \|x\||^* \leq \a|x|, \) and we define \( M_r^* = \int_{S^{n-1}} \|x\||^* d\mu(x). \)

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Let \( K = \{ x \in X: \|x\| \leq 1 \} \), \( K^* = \{ x \in X: \|x\|^* \leq 1 \} \), and \( D = \{ x \in X: |x| \leq 1 \} \). We consider also the usual \((n\text{-dimensional})\) Lebesgue measure \((\text{Vol}_n)\) on \( R^n \) normalized (for example) so that the induced measure on \( S^{n-1} \) coincides with \( \mu(x) \); that is,
\[
\text{Vol}_n(D) = (1/n)\mu(S^{n-1}) = 1/n.
\]

We will use the following geometrical inequalities:

1. \((\text{Vol}_n K / \text{Vol}_n D)^{1/n} \leq M_r^* \) (the Urysohn inequality \([U]\)),
2. \( \text{Vol}_n K \cdot \text{Vol}_n K^* \leq (\text{Vol}_n D)^2 \) (the Santalo inequality \([S]\)).

Also define \( M_r \cdot M_r^* = M \).

3. We prove the following proposition (see also \([M_2]\)).

**Proposition.** For every \( \lambda, 0 < \lambda < 1 \), there exists a subspace \( E \subset X, \dim E \geq \lambda n \), such that \( E^* \) contains a subspace \( F \subset E^* \), \( \dim F = k \geq \lambda^2 n \), such that
\[
d(F, l_2^k) \leq [C_1(M + 1)]^{2A(1-\lambda)^2},
\]
where \( C_1 \) is an absolute constant (say \( \sim 8\pi \)).

**Proof.** We start with a general argument valid for an arbitrary Euclidean norm \( | \cdot | \) on \( R^n \). This norm will be defined in §§5 and 6. We introduce an additional norm \( \| \cdot \| \) on \( X \) such that
\[
K_1 = \{ x \in X: \|x\|_1 \leq 1 \} = \text{Conv}(K \cup D).
\]
Then \( K_1^* = \{ x \in X: \|x\|^*_1 \leq 1 \} = K^* \cap D \) (i.e., \( \|x\|^*_1 = \max(\|x\|^*, |x|) \)). Therefore,
\[
M_r \cdot M_r^* = \int_{S^{n-1}} \|x\|_1 d\mu(x) \leq M_r^* + 1.
\]
Since \( \|x\|_1 \leq |x| \), the so-called volume ratio of the pair \((K_1; D)\) is
\[
\nu(K_1) \overset{\text{def}}{=} (\text{Vol}_n K_1 / \text{Vol}_n D)^{1/n} \leq M_r^* + 1 \overset{\text{def}}{=} A
\]
(by (1)). Next we use the following statement, which is an immediate consequence of the technique of Szarek’s proof \([Sz]\) of Kashin’s theorem \([K]\) (for details see \([M_2]\)).

**4. Statement.** Let \( \nu(K_1) \leq A \). Fix \( 0 < \lambda < 1 \). Then for any \( k \leq \lambda n \) there exists a subspace \( E \), \( \dim E = k \), such that
\[
\frac{1}{2}(2\pi A)^{-\theta} |x| \leq \|x\|_1 \leq \|x\|,
\]
where \( \theta = 1/(1-\lambda) \). The normalized Haar measure \( \nu_{n,k} \) of such subspaces in the Grassmann manifold \( G_{n,k} \) of \( k\)-dimensional subspaces of \( R^n \) is at least \( 1 - 1/2^{n-1} \).

5. We return to the proof of Proposition 3 and consider the normalization \( M_r = 1 \).
Let \( a = \{ x \in S: \|x\| \leq 2 \} \). Then obviously \( \mu(a) \geq 1/2 \). Fix \( \delta > 0 \). Let \( B = \{ \xi \in G_{n,k}: \mu_\xi(a \cap \xi) \geq \delta \} \). Again, obviously, \( \nu_{n,k}(B) = \gamma \geq (\frac{1}{2} - \delta)/(1 - \delta) \). Then for any \( \xi \in B \),
\[
\frac{\text{Vol}_\xi(K \cap \xi)}{\text{Vol}_\xi(D \cap \xi)} = \int_{S \cap \xi} \frac{1}{\|x\|^k} d\mu(x) \geq \frac{1}{2^k} \delta.
\]
Choose \( \delta = 1/4 \) and so \( \gamma \geq 1/3 \).

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Therefore, there exists a subspace $E_0$ (and actually a large measure of subspaces) as in Statement 4 such that

\[ \text{Vol}_k(K \cap E_0)/\text{Vol}_k(D(E_0)) \geq 1/2^k \cdot 1/4, \]

where $k = \dim E_0$. Consider now $E_0^*$. Then for any $x \in E_0^*$, $\|x\|^* \leq (2\pi A)^\theta |x|$, and by (3) and the Santalo inequality (2),

\[ \text{Vol}_k(K \cap E_0)^*/\text{Vol}_k(D(E_0)) \leq 4 \cdot 2^k. \]

Therefore, introducing a new norm on $E_0$: $\|x\|_2 = \|x\|^*/(2\pi A)^\theta$, we have that $K_2 = \{ x : \|x\|_2 \leq 1 \}$ has the volume ratio

\[ \text{vr}(K_2) \leq A_1 = 4(2\pi A)^\theta \cdot 4^{1/k} \quad (\text{and } \|x\|_2 \leq |x|). \]

So we may use Statement 4 one more time for the norm $\| \cdot \|_2$ to finish the proof of Proposition 3.

6. Proposition 3 contains a number $M$ which depends on the choice of a euclidean norm in $R^n$. It is known [F, T.] that for every $(X, \| \cdot \|)$ there exists a euclidean norm $| \cdot |$ such that $M + 1 \leq c_2||\text{Rad}_X||$, where $c_2$ is an absolute constant and $||\text{Rad}_X||$ is the norm of the so-called Rademacher projection of $L_2(X)$ onto Rad $X$, which, as G. Pisier [P] has proved, may be estimated by

\[ ||\text{Rad}_X|| \leq c_3 \ln (d_X + 1), \]

where $c_3$ is an absolute constant and $d_X = d(X, l_2^n) \leq \sqrt{n}$. Therefore, in particular,

\[ ||\text{Rad}_X|| \leq c_3 \ln (n + 1). \]

7. Using 6, we may write in Proposition 3 that

\[ d(F, l_2^n) \leq \left[ c \ln (d_X + 1) \right]^{1/(1 - \lambda)^2} \leq \left[ c \ln (n + 1) \right]^{1/(1 - \lambda)^2}, \]

where $c$ is a universal constant. We now use Proposition 3 and (5) consecutively many times starting with $\lambda_1$, obtaining a space $F_1$ (as in Proposition 3), $\dim F_1 = k_1 \geq \lambda_1^n$ with

\[ d_1 = d(F_1, l_2^{k_1}) \leq \left[ c \ln (n + 1) \right]^{1/(1 - \lambda_1)^2}. \]

For the second step we apply the same Proposition 3 to space $F_1$ (instead of $X$) with $\lambda_2$ and obtain a space $F_2$, $\dim F_2 = k_2 \geq (\lambda_2 \lambda_1)^n$ with

\[ d_2 = d(F_2, l_2^{k_2}) \leq \left[ c \ln (d_1 + 1) \right]^{1/(1 - \lambda_2)^2}, \]

and so on.

It remains to state how we choose $\lambda_t, t = 1, 2, \ldots$. The notations $\ln^{(1)} A$ will be used for the $t$-times iterated logarithm of $A$ (so $\ln^{(2)} A = \ln \ln A$) if for any $k \leq t$, the $k$-iterated logarithm of $A$ is at least 2 and just 2 in the opposite case. With such an agreement we write, in (5), $\ln^{(1)} d_X (= \ln d_X)$ and $\ln^{(1)} n$ instead of $\ln(d_X + 1)$ and $\ln(n + 1)$.

For every $t \geq 1$, take $\lambda_t = 1 - \sqrt{6}/\ln^{(t+1)} n$, and we obtain, by using Proposition 3 $t$-times, a space $F_t$,

\[ \dim F_t = k_t \geq \prod_{i=1}^t \left( 1 - \sqrt{6}/\ln^{(t+1)} n \right)^2 n \]
and
\[ d_t = d(F_t, l^2_{F_t}) \leq (c \ln d_{t-1})^{(\ln^{(t+1)} n)^2/6}. \]

We assume now that \( c < \ln d_{t-1} \), and we just stop our iteration in the opposite case, \( d_{t-1} \leq e^c \). Therefore,
\[ d_t \leq (\ln d_{t-1})^{(\ln^{(t+1)} n)^2/3} \]

and
\[ \ln d_t \leq \left( (\ln^{(t+1)} n)^2/3 \right) \ln^{(2)} d_{t-1}. \]

Now,
\[ \ln d_1 \leq \left( \ln^{(2)} n \right)^3/3 \leq \left( \ln^{(2)} n \right)^3, \quad \ln d_2 \leq \left( \ln^{(3)} n \right)^3, \]

and, in general,
\[ \ln d_t \leq \left( \ln^{(t+1)} n \right)^3. \]

Now take \( d \) from the statement of Theorem 1. By Remark 1 we may assume \( d \) large enough; so let \( d > e^c \) and \( a = (\ln \ln d)/3 > \sqrt{6} \cdot 2 \). We stop our iteration procedure for \( r \) such that for the last time, \( \ln^{(r+1)} n \geq a \) (i.e., \( e^a > \ln^{(r+1)} n \)), which implies, by (8) and (7), \( d_t \leq \exp(e^{3a}) = d \). Of course, the iteration could have stopped before because of the first condition if, for some \( j < t \), \( d_j \leq e^c < d \). Therefore, in both cases we have found a space \( F_{j+1} \leq F_{j+2} \leq \cdots \leq F_t \leq F_{t+1} \) such that \( d(F_{j+1}, l^2_{F_{j+1}}) \leq d \). It remains to estimate \( \dim F_t = k_t \), using (6). On the last step of the iteration we have \( \ln^{(r+1)} n \geq a \) (\( > 2\sqrt{6} \)), and, therefore, \( \lambda_t \geq (1 - \sqrt{6}/a) \), \( \lambda_{t-1} \geq (1 - \sqrt{6}/e^a) \), and so on; so it is enough to estimate from below the infinite product
\[ (1 - \sqrt{6}/a) \cdot (1 - \sqrt{6}/e^a) \cdot \cdots \]

\[ \prod_{t=0}^{\infty} \left( 1 - \sqrt{6}/a^{2^t} \right) = f(a) \to 1 \]

if \( a \to \infty \), and, therefore, we prove the principal part of Theorem 1. It is also clear that the main part of this product is the first term \( f(a) \approx 1 - \sqrt{6}/a = 1 - 3\sqrt{6}/\ln \ln d \).

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Added in proof. I have now obtained the following better estimate for the function \( \lambda(d) \) in Theorem 1:
\[ \lambda(d) \geq 1 - c \sqrt{\frac{\log d}{d}} \quad \text{for large } d. \]

References


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