ON SURFACES IN $\mathbb{R}^4$

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Abstract. We provide answers (Theorem C) to some questions concerning surfaces in $\mathbb{R}^4$ and maps into the quadric $Q_2$ raised by D. Hoffman and R. Osserman.

Let $S$ be an oriented surface immersed in $\mathbb{R}^4$. The Gauss map of $S$ is the map $G$ of $S$ into $G(2,4)$, the Grassmannian of oriented two-planes in $\mathbb{R}^4$, given by $G(p) = T_p S$. $G(2,4)$ can be identified with $Q_2$, the complex quadric in $\mathbb{C}P^3$, and in turn $Q_2$ is biholomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$. If we give $\mathbb{C}P^3$ the Fubini-Study metric of constant holomorphic sectional curvature 2, then the induced metric on $Q_2$ is given by

$$2|dw_1|^2/(1 + |w_1|^2)^2 + 2|dw_2|^2/(1 + |w_2|^2)^2,$$

where $(w_1, w_2)$ are coordinates on $\mathbb{C} \times \mathbb{C}$, viewed as local coordinates on $\mathbb{C}P^1 \times \mathbb{C}P^1$ [1]. The metric $2|dw|^2/(1 + |w|^2)^2$ is the metric on $\mathbb{C}$ induced by the map of $\mathbb{C}$ onto $S^2(1/\sqrt{2}) \subset \mathbb{R}^3$ given by $w \mapsto \sigma^{-1}(\sqrt{2} w)$, where $\sigma^{-1}$ is inverse stereographic projection (with the sphere sitting on the $xy$-plane). Thus, $Q_2$ is isometric to $S^2(1/\sqrt{2}) \times S^2(1/\sqrt{2})$. In particular, if $z$ is a local conformal parameter on $S$, then any map $G$ of $S$ into $Q_2$ splits into a pair of maps $G = (f_1(z), f_2(z))$, where $w_i = f_i(z)$ as above.

Now define the following quantities on $S$ for $i = 1, 2$:

$$F_i := \frac{f_i\bar{z}}{1 + |f_i|^2}, \quad T_i(z) = \left[ \frac{(f_i)_{zz}}{(f_i)_z} - \frac{2\bar{f}_zf_{iz}}{1 + |f_i|^2} \right] \text{ where } f_{iz} \neq 0$$

with the usual $z$ and $\bar{z}$ derivative notation. The following results are from [1, 2].

**Theorem A.** For the Gauss map $G$ of an oriented surface $S$ immersed in $\mathbb{R}^4$, write $G = (f_1(z), f_2(z))$ as above. Then we necessarily have

(1) $|F_1| \equiv |F_2|,$

and

(2) $\text{Im} \{T_1 + T_2\} \equiv 0.$

**Theorem B.** Let $S_0$ be a simply connected Riemann surface (here and subsequently), let $G = (f_1(z), f_2(z))$ be some map of $S_0$ into $Q_2$, and define $F_i$ and $T_i$ as before, where $z$ is a conformal parameter on $S_0$.

(i) If $F_1 = F_2 \equiv 0$, then $G$ is the Gauss map of a minimal surface in $\mathbb{R}^4$, provided $S_0$ is not compact.
(ii) If $F_1$, $F_2$ are never zero, then $G$ is the Gauss map of a surface $S$ in $\mathbb{R}^4$ given by a conformal immersion of $S_0$ if and only if

(1) $|F_1| = |F_2|$, 

and

(2) $\text{Im}\{T_1 + T_2\} = 0$.

Furthermore, in this case $S$ is uniquely determined up to translation and homothety of $\mathbb{R}^4$.

Let (1') denote the condition that $F_1$, $F_2$ are never zero (i.e., $f_1\bar{z}$ and $f_2\bar{z}$ are never zero) and $|F_1| = |F_2|$. A special class of maps which satisfy (2) are harmonic maps, i.e., those $f(z)$'s such that

$$L(f) := f_{\bar{z}z} - 2\bar{f}f_{\bar{z}z}/(1 + |f|^2) = 0.$$ 

In particular, if

(3) $L(f_i) = 0, \quad i = 1, 2,$

then (2) is automatically satisfied. Condition (3) is simply that the map $G: S_0 \to \mathbb{Q}_2$ is harmonic. A theorem of Ruh and Vilms [4] asserts that the Gauss map of a submanifold of $\mathbb{R}^n$ is harmonic if and only if the submanifold has parallel mean curvature. Combining this with Theorem B we now have the following observation:

A map $G: S_0 \to \mathbb{Q}_2$ is the Gauss map of a conformal immersion with parallel (nonzero) mean curvature in $\mathbb{R}^4$ if and only if (1') and (3) hold.

Finally, an interesting subclass of surfaces of parallel mean curvature in $\mathbb{R}^4$ are minimal surfaces in some $S^3(r)$. Hoffman and Osserman also prove the following

**Proposition.** A map $G: S_0 \to \mathbb{Q}_2$ is the Gauss map of a conformal minimal immersion of $S_0$ into some $S^3(r)$ (viewed as sitting in $\mathbb{R}^4$) if and only if (1) and (3) are satisfied, as well as the following

(4) $f_{1z}/f_{1\bar{z}} = f_{2z}/f_{2\bar{z}}$.

In view of these results, the following questions present themselves [1, 2]: Given a map from $S_0$ into $S^2(1/\sqrt{2})$, represented locally by $f_1(z)$ as above, does there exist a map from $S_0$ into $S^2(1/\sqrt{2})$, represented by $f_2(z)$, such that the pair $(f_1(z), f_2(z))$ satisfies

Q1. (1') and (2)? Suppose $f_1$ satisfies $L(f_1) = 0$. Does there exist $f_2(z)$ such that the pair $(f_1, f_2)$ satisfies

Q2. (1') and (3), or

Q3. (1'), (3) and (4)?

An affirmative answer to Q1 (Q2) would mean that the pair $(f_1, f_2)$ is the Gauss map of a conformal immersion (with parallel nonzero mean curvature) of $S_0$ in $\mathbb{R}^4$, while an affirmative answer to Q3 would mean that the pair $(f_1, f_2)$ is the Gauss map of a conformal minimal immersion of $S_0$ into some $S^3(r)$ (viewed as sitting in $\mathbb{R}^4$).
We answer Q2 and Q3 affirmatively in Theorem C. While this provides an affirmative answer to Q1 under the special assumption of (3) ($\Rightarrow$ (2)), we do not know the answer to Q1 in general.

**Theorem C.** Given a map from $S_0$, not conformally equivalent to $S^2$, into $S^2(1/\sqrt{2})$, written as $f_1(z)$ as above, such that $f_{1\bar{z}}$ is never zero, and $L(f_1) = 0$, there exists a one-parameter family of maps of $S_0$ into $S^2(1/\sqrt{2})$, written as $f_\theta(z)$, such that the pair $(f_1, f_\theta)$ satisfies (1) and (3). Furthermore, there is a unique $\theta_0$ such that the pair $(f_1, f_{\theta_0})$ also satisfies (4). If $S_0$ is conformally equivalent to $S^2$, $f_2 = f_1$ is the only possibility for even (1) and (3).

**Remark.** The idea of the proof is to regard $f_1$ as the Gauss map of a surface $S$ of constant (nonzero) mean curvature in $\mathbb{R}^3$. The $f_\theta$'s are the Gauss maps of the associated family $S_\theta$, $0 \leq \theta \leq 2\pi$, to $S$. It turns out that condition (4) is then satisfied exactly for the surface $S_\theta$:

**Proof of Theorem C.** We regard $f_1(z)$ as the representation of a map of $S_0$ with $S^2(1)$ as follows: Let $\sigma(\sigma')$ be stereographic projection of $S^2(1/\sqrt{2}) (S^2(1))$ onto $\mathbb{C}$, and consider the transformation $\mathbb{C} \rightarrow \mathbb{C}$ by

$$
\phi(w) = \frac{1}{2} \left( \sigma'\left(\frac{1}{\sqrt{2}} \sigma^{-1}(\sqrt{2} w)\right) \right).
$$

$\phi$ is just the identity map on $\mathbb{C}$, so $\phi(f_1) = f_1$. Replacing $f_1$ by $\bar{f}_1 = \sqrt{2} (\sigma^{-1}(\sqrt{2} f_1)) \in S^2(1)$, and then representing $\bar{f}_1$ by $\frac{1}{2} \sigma(\bar{f}_1)$, we see that we may regard $f_1$ as a map into $\mathbb{C}P^1$, with the metric $4|dw|^2/(1 + |w|^2)^2$ of constant curvature 1. Thus it suffices to prove Theorem C with $S^2(1/\sqrt{2})$ replaced by $S^2(1)$. Now the conditions $f_{1\bar{z}} \neq 0$, $L(f_1) = 0$ mean that $f_1$ is a harmonic, nowhere anticonformal map of $S_0$ into $S^2 = S^2(1)$. From Hoffman-Osserman [1] and Kenmotsu [3], this guarantees that $f_1$ is the Gauss map of a conformal immersion $X$ of $S_0$ into $\mathbb{R}^3$ with constant nonzero mean curvature. If we specify that $X(S_0)$ have constant mean curvature 1, then this determines $\mathcal{S}_0 = X(S_0)$ up to translation in $\mathbb{R}^3$. If $S^0$ is conformally equivalent to $S^2$, then $\mathcal{S}_0$ is the standard unit sphere, and any $f_2$: $S_0 \rightarrow S^2(1)$ satisfying (1) and (3) must come from the same $X$ (up to translation of $\mathbb{R}^3$). For $S_0$ not conformally $S^2$, in the (global) isothermal parameter $z$, the metric induced on $S_0$ is $4 F_0|dz|^2$ [3], where we have relabelled $f_1$ as $f_0$. Now consider the associate family $\mathcal{S}_\theta [5]$ to $\mathcal{S}_0 (0 \leq \theta < 2\pi)$. Then the Gauss maps $f_\theta$ of $\mathcal{S}_\theta$ satisfy (3), since each $\mathcal{S}_\theta$ has constant mean curvature. Since the metric on $S_0$ inherited from $\mathcal{S}_\theta$ is given by

$$
4|f_{\theta z}|^2/(1 + |f_{\theta z}|^2)^2 |dz|^2,
$$

and since $\mathcal{S}_\theta$ is isometric to $\mathcal{S}_0$, we also have condition (1) satisfied for the pair $(f_0, f_\theta)$. Let $\beta_\theta$ be the second fundamental form of $\mathcal{S}_\theta$. Then from formula 5.3 of [3], we have

$$
\frac{1}{F_\theta} \left\{ \frac{\beta_{\theta 11} - \beta_{\theta 22}}{2} - i \beta_{\theta 12} \right\} = \frac{f_{\theta z}}{f_{\theta \bar{z}}}.
$$
From this we see that condition (4), in the presence of (1), is equivalent to

\[ \beta_{11}^0 - \beta_{22}^0 = \beta_{22}^\theta - \beta_{11}^\theta, \quad \beta_{12}^0 = -\beta_{12}^\theta. \]  

Finally, since

\[ \beta_{11}^\theta = \cos \theta (\beta_{11}^0 - F_0) + \sin \theta \beta_{12}^0 + F_0, \]

\[ \beta_{22}^\theta = -\cos \theta (\beta_{11}^0 - F_0) - \sin \theta \beta_{12}^0 + F_0, \]

\[ \beta_{12}^\theta = \cos \theta \beta_{12}^0 - \sin \theta (\beta_{11}^0 - F_0) \]

and \( \beta_{11}^0 + \beta_{22}^0 = 2F_0 [5] \), we see that condition (4) (cf. (6)) is equivalent to \( \theta = \pi \). Q.E.D.

REFERENCES

2. ______, The Gauss map of surfaces in \( \mathbb{R}^3 \) and \( \mathbb{R}^4 \) (to appear).

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