ON THE EXISTENCE OF UNIFORMLY CONSISTENT ESTIMATES

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Abstract. Let be a family of probability measures on \((\mathcal{X}, \mathcal{A})\) and \(U\) the uniform structure defined by vicinities of the form

\[
\left\{ (P, Q) : \sup_{1 \leq i \leq K} |P^n(A_{i,n}) - Q^n(A_{i,n})| < \varepsilon \right\},
\]

where \(P^n\) is the product measure on \((\mathcal{X}^n, \mathcal{A}^n)\), \(A_{i,n} \in \mathcal{A}^n\), \(\varepsilon > 0\), \(n \wedge K \geq 1\). Let \(\phi^*: (\mathcal{M}, U) \to (\phi^*(\mathcal{M}), d)\), where

\[
d(\phi^*(P), \phi^*(Q)) = \|P - Q\|_1 \leq 2 \sup_{A \in \mathcal{A}^n} |P(A) - Q(A)|.
\]

We consider the case where the space of measures \(M\) is \(L_1\) separable and relate the existence of uniformly consistent estimates for \(\phi^*(P)\) with uniform continuity of \(\phi^*\) and \(L_1\)-total boundedness of \(M\).

1. Introduction, notation and definitions. Let \((\mathcal{X}, \mathcal{A})\) be a space with a \(\sigma\)-field, let \(\mathcal{M}\) be a family of probability measures on \(\mathcal{A}\), \((\mathcal{X}^n, \mathcal{A}^n)\) the \(n\)th product space and \(\sigma\)-field, and let \(X_1, \ldots, X_n\) be independent identically distributed observations according to some measure \(P \in \mathcal{M}\), \(P^n\) being the \(n\)th product measure. Let \(\Theta\) be a topological space which is homeomorphic to a subset of \([0,1]^\infty\), \(h\) being the homeomorphism, and let \(\rho\) be a metric on \([0,1]^\infty\) of the form

\[
\rho(x, y) = \sum_{m=1}^{\infty} 2^{-m}|x_m - y_m|,
\]

where \(x_m, y_m\) are coordinates of \(x, y\) respectively. Let \(\phi^*: P \to \phi^*(P)\) be a function defined on \(\mathcal{M}\) with values in \(\Theta\), and \(\phi = h \circ \phi^*, \phi: \mathcal{M} \to ([0,1]^\infty, \rho)\).

LeCam and Schwartz [1960] gave necessary and sufficient conditions for the existence of uniformly consistent estimates of \(\phi(P)\) in terms of the uniform continuity of \(\phi\) with respect to a uniform structure \(U = \bigcup_n U_n\), where each \(U_n\) consists of vicinities of the form \(\left\{ (P, Q) : \sup_{1 \leq j \leq l} |\int f_j dP^n - \int f_j dQ^n| < 1\right\}\). The same uniform structure has been used by Pfanzagl [1968] and Moché [1977]. Under the above set-up we will explain why it is natural to consider the uniform structure \(U\) and we...
will offer a theorem on the existence of estimates. Relaxing the condition of homeomorphism of \(\Theta\) with a subset of \([0,1]^{\infty}\) and assuming only that \((\phi^*(\mathcal{M}), d)\) is separable when metrized with the total variation \((L_1)\) norm between the measures, i.e. \(d(\phi^*(P), \phi^*(Q)) = \|P - Q\|_{L_1} = 2\sup\{|P(A) - Q(A)|; A \in \mathcal{A}\}\), we offer a theorem of the same type in the form of equivalent propositions.

**Definition.** We shall say \(\phi\) is uniformly consistently estimable if there is a sequence \(T_n\) of measurable functions from \((\mathcal{X}^n, \mathcal{A}^n)\) to \([0,1]^{\infty}, \rho\) such that for every \(\varepsilon > 0\), \(\sup\{ P^n[\rho(T_n, \phi(P)) > \varepsilon]; P \in \mathcal{M} \} \to 0\) as \(n\) tends to infinity.

**Remark.** One can easily see that since \(\rho\) is bounded, the above definition is equivalent to \(\sup\{ E_P \rho(T_n, \phi(P)); P \in \mathcal{M} \} \to 0\) as \(n\) tends to infinity.

Recall from topology the following

**Definition.** Let \((Y, d)\) be a metric space with a metric \(d\). We say \(Y\) is \(d\)-totally bounded if for every \(\varepsilon > 0\) there are \(y_1, \ldots, y_{n(\varepsilon)}\) elements of \(Y\) such that

\[
Y = \bigcup_{i=1}^{n(\varepsilon)} \{ y: d(y, y_i) < \varepsilon \}.
\]

In our effort to relax the hypothesis of homeomorphism of \(\Theta\) with \([0,1]^{\infty}\) our estimator will take values in an abstract space. Terminology and results under this set-up concerning measurability (strong and weak), convergence (almost sure and almost uniform) and Bochner integration can be found in Hille and Phillips [1957, §§3.5 and 3.7]. For the notion of uniform structure the reader is referred to Choquet [1969, Chapter 2, §5]. For a deep insight on consistency questions, we suggest the remarkable paper of Kraft [1955].


**Lemma 1.** Assume \(\phi\) is uniformly consistently estimable by \(T_n\). Then for every \(\varepsilon > 0\) there exist positive integers \(n = n(\varepsilon), K = K(\varepsilon)\) and \(l = l(\varepsilon)\), and measurable sets \(B_1^n, \ldots, B_l^n\) of the \(n\)th product \(\sigma\)-field \(\mathcal{A}^n\) such that

\[
\rho(\phi(P), \phi(Q)) \leq \varepsilon + K \cdot \sup\{|P^n(B_j^n) - Q^n(B_i^n)|; 1 \leq j \leq l(\varepsilon)\}.
\]

**Proof.** We will use the triangular inequality and then Jensen’s inequality,

\[
\rho(\phi(P), \phi(Q)) \leq \rho(\phi(P), E_{P^\ast}T_n) + \rho( E_{P^\ast}T_n, E_{Q^\ast}T_n) + \rho( E_{Q^\ast}T_n, \phi(Q))
\]

\[
\leq E_{P^\ast}\rho(\phi(P), T_n) + \rho( E_{P^\ast}T_n, E_{Q^\ast}T_n) + E_{Q^\ast}\rho(T_n, \phi(Q))
\]

\[
\leq 2 \cdot \frac{\varepsilon}{4} + \rho( E_{P^\ast}T_n, E_{Q^\ast}T_n)
\]

for every \(n \geq n(\varepsilon)\) by uniform convergence of \(T_n\) to \(\phi(P)\). For the rest of the proof let \(n = n(\varepsilon)\).

Consider

\[
\rho( E_{P^\ast}T_n, E_{Q^\ast}T_n) = \sum_{m=1}^{\infty} 2^{-m}|E_{P^\ast}(T_n)_m - E_{Q^\ast}(T_n)_m|,
\]
where \((T_n)_m\) are the coordinates of \(T_n\), \(1 \leq m < \infty\). By assumption \(|E_{p^*}(T_n)_m - E_{Q^*}(T_n)_m| \leq 2\), so there is \(m_0 = m_0(\varepsilon) > 0\) such that

\[
\sum_{m=m_0+1}^{\infty} 2^{-m}|E_{p^*}(T_n)_m - E_{Q^*}(T_n)_m| \leq \sum_{m=m_0+1}^{\infty} 2^{-m+1} < \frac{\varepsilon}{4}.
\]

So now (2) becomes

(3) \[
\rho\left(E_{p^*}T_n, E_{Q^*}T_n\right) \leq \frac{\varepsilon}{4} + \sum_{m=1}^{m_0} 2^{-m}|E_{p^*}(T_n)_m - E_{Q^*}(T_n)_m|.
\]

Consider now \((T_n)_m\) for some \(m \in \{1, \ldots, m_0\}\). Since \(0 \leq (T_n)_m \leq 1\) for \(\varepsilon/8\) there exists a simple function of the form \(\sum_{i=1}^{l_m(\varepsilon)} a_{i,m} I_{A^n_{i,m}}\) such that

\[
\sup \left\{ \left| (T_n(x_1, \ldots, x_n))_m - \sum_{i=1}^{l_m(\varepsilon)} a_{i,m} I_{A^n_{i,m}} \right| ; (x_1, \ldots, x_n) \in \mathbb{R}^n \right\} < \frac{\varepsilon}{8}.
\]

The \(m\)th term of the sum in (3) becomes

\[
|E_{p^*}(T_n)_m - E_{Q^*}(T_n)_m| \leq 2\varepsilon/8 + \sum_{i=1}^{l_m(\varepsilon)} a_{i,m} |P^n(A^n_{i,m}) - Q^n(A^n_{i,m})|
\]

\[
\leq \frac{\varepsilon}{4} + \left( \sum_{i=1}^{l_m(\varepsilon)} |a_{i,m}| \right) \cdot \sup \left\{ |P^n(A^n_{i,m}) - Q^n(A^n_{i,m})| ; 1 \leq i \leq l_m(\varepsilon) \right\}.
\]

Letting \(K_m(\varepsilon) = \sum_{i=1}^{l_m(\varepsilon)} |a_{i,m}|\) and repeating the same argument for all \(m \in \{1, \ldots, m_0\}\) in (3) we get

\[
\rho\left(E_{p^*}T_n, E_{Q^*}T_n\right) \leq \frac{\varepsilon}{4} + \sum_{m=1}^{m_0} 2^{-m} \cdot \frac{\varepsilon}{4}
\]

\[
+ \sum_{m=1}^{m_0} 2^{-m} \cdot K_m(\varepsilon) \cdot \sup \left\{ |P^n(A^n_{i,m}) - Q^n(A^n_{i,m})| ; 1 \leq i \leq l_m(\varepsilon) \right\}
\]

\[
\leq \frac{\varepsilon}{2} + \left( \sum_{m=1}^{m_0} 2^{-m} K_m(\varepsilon) \right) \cdot \sup \left\{ |P^n(B^n_j) - Q^n(B^n_j)| ; 1 \leq j \leq \sum_{m=1}^{m_0} l_m(\varepsilon) \right\}
\]

By letting \(K(\varepsilon) = \sum_{m=1}^{m_0} 2^{-m} K_m(\varepsilon)\), \(l(\varepsilon) = \sum_{m=1}^{m_0} l_m(\varepsilon)\) and replacing in (1) we get

\[
\rho(\phi(P), \phi(Q)) \leq \varepsilon + K \cdot \sup \left\{ |P^n(B^n_j) - Q^n(B^n_j)| ; 1 \leq j \leq l(\varepsilon) \right\}.
\]

Q.E.D.

The above lemma shows that giving \(\mathcal{M}\) the uniform structure consisting of vicinities of the form \(\{(P, Q): \sup \{|P^n(A^n_i) - Q^n(A^n_i)| ; 1 \leq i \leq l\} < \varepsilon\}\) for \(l \in N^+\), \(n \in N^+, \varepsilon > 0\), is the natural way for associating uniform convergence in probability of \(T_n\) with the uniform continuity of \(\phi: \mathcal{M} \to ([0,1]^\infty, \rho)\).

It is now easy to see that these vicinities give rise to the same uniform structure \(U\) as those in the introduction.
We offer the following

**Lemma 2.** Let $\phi: (\mathcal{M}, U) \to ([0, 1]^\infty, \rho)$. The following are equivalent:
1. $\phi$ is uniformly continuous.
2. For every $\varepsilon > 0$ there exists $K(\varepsilon), m(\varepsilon), n(\varepsilon)$ all positive, and $B_i^{m(\varepsilon)}, \ldots, B_n^{m(\varepsilon)}$ elements of the $m(\varepsilon)$-product $\sigma$-field $\mathcal{M}^{m(\varepsilon)}$ such that
   \[ \rho(\phi(P), \phi(Q)) \leq \varepsilon + K(\varepsilon) \cdot \sup \{|P^{m(\varepsilon)}(B_i^{m(\varepsilon)}) - Q^{m(\varepsilon)}(B_i^{m(\varepsilon)})|; 1 \leq i \leq n(\varepsilon)\}. \]

**Proof.** $(2) \Rightarrow (1)$. Obvious.

$(1) \Rightarrow (2)$. Since $\phi$ is uniformly continuous, for every $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$, $m = m(\varepsilon) > 0$, $n = n(\varepsilon) > 0$ and sets $B_1^{m(\varepsilon)}, \ldots, B_n^{m(\varepsilon)}$ in $\mathcal{M}^{m(\varepsilon)}$ such that if
   \[ \sup \{|P^{m(\varepsilon)}(B_i^{m(\varepsilon)}) - Q^{m(\varepsilon)}(B_i^{m(\varepsilon)})|; 1 \leq i \leq n(\varepsilon)\} < \delta(\varepsilon), \]
then $\rho(\phi(P), \phi(Q)) < \varepsilon$.

From this, for every $P, Q$ such that $\rho(\phi(P), \phi(Q)) \geq \varepsilon$ it is necessary that
   \[ \sup \{|P^{m(\varepsilon)}(B_i^{m(\varepsilon)}) - Q^{m(\varepsilon)}(B_i^{m(\varepsilon)})|; 1 \leq i \leq n\} \geq \delta(\varepsilon). \]

Going to the statement it is required to prove: Note that for every $\varepsilon > 0$ either
   \[ \rho(\phi(P), \phi(Q)) < \varepsilon \text{ or } \rho(\phi(P), \phi(Q)) \geq \varepsilon. \]
For the second case we have then
   \[ \rho(\phi(P), \phi(Q)) \leq \rho(\phi(P), \phi(Q)) \cdot \left[ \inf \left\{ \sup \{|P^{m(\varepsilon)}(B_i^{m(\varepsilon)}) - Q^{m(\varepsilon)}(B_i^{m(\varepsilon)})|; 1 \leq i \leq n(\varepsilon)\}; \right. \right. \]
   \[ \left. \left. P, Q: \rho(\phi(P), \phi(Q)) > \varepsilon \right\} \right]^{-1}, \]
   \[ \cdot \sup \{|P^{m(\varepsilon)}(B_i^{m(\varepsilon)}) - Q^{m(\varepsilon)}(B_i^{m(\varepsilon)})|; 1 \leq i \leq n\} \]
   \[ \leq 2[\delta(\varepsilon)]^{-1} \sup \{|P^{m(\varepsilon)}(B_i^{m(\varepsilon)}) - Q^{m(\varepsilon)}(B_i^{m(\varepsilon)})|; 1 \leq i \leq n\}. \]
Let $2[\delta(\varepsilon)]^{-1} = K(\varepsilon)$. So finally for all $P, Q$
   \[ \rho(\phi(P), \phi(Q)) \leq \varepsilon + K(\varepsilon) \cdot \sup \{|P^{m(\varepsilon)}(B_i^{m(\varepsilon)}) - Q^{m(\varepsilon)}(B_i^{m(\varepsilon)})|; 1 \leq i \leq n\}. \]

We offer

**Theorem 1.** Let $\phi: (\mathcal{M}, U) \to ([0, 1]^\infty, \rho)$ with $U$ defined as above. The following statements are equivalent:
1. $\phi$ is uniformly continuous.
2. There exists a uniformly consistent estimator of $\phi(P)$.

**Proof.** $(1) \Rightarrow (2)$. Given in LeCam and Schwartz [1960, Theorem 1].

$(2) \Rightarrow (1)$. Corollary of Lemmas 1 and 2.

3. **The main result.** Consider now the case $\phi^*: (\mathcal{M}, U) \to (\phi^*(\mathcal{M}), \| \cdot \|_{L_1})$, as in the introduction, with $\phi^*(\mathcal{M})$ separable when metrized with the $L_1$-distance between measures. We will give our result in a series of lemmas.

**Lemma 3.** The uniform structure $U$ is precompact (in the sense that for any vicinity of the form $\{P, Q: \sup \{|P^{m}(A_i^{m}) - Q^{m}(A_i^{m})|; 1 \leq i \leq n\} < \varepsilon\}$ there exist $P_1, \ldots, P_L$ in $\mathcal{M}$ such that $\mathcal{M} = \bigcup_{k=1}^{L} \{P: \sup \{|P^{m}(A_i^{m}) - P_k^{m}(A_i^{m})|; 1 \leq i \leq n\} < \varepsilon\}$).

**Proof.** Given in LeCam and Schwartz [1960, p. 142].
Lemma 4. Let \( \phi^* : (\mathcal{M}, U) \to (\phi^*(\mathcal{M}), \| \cdot \|_{L_1}) \). The following propositions are equivalent:

(A) \( \phi^* \) is uniformly continuous.

(B) \( \mathcal{M} \) is \( L_1 \)-totally bounded.

Proof. (B) \( \Rightarrow \) (A). To prove \( \phi^* \) is uniformly continuous, it suffices to prove that for every \( \varepsilon > 0 \) there exist \( \delta(\varepsilon) \) and \( B_1, \ldots, B_n \) sets in \( \mathcal{A}^m \) such that for every \( (P, Q) \in V_{\delta(\varepsilon), m, B_1, \ldots, B_n} = \{ (\tilde{P}, \tilde{Q}) : \sup_{1 \leq i \leq n} |\tilde{P}^m(B_i) - \tilde{Q}^m(B_i)| < \delta(\varepsilon) \} \) implies \( \| \phi^*(P) - \phi^*(Q) \| = \| P - Q \|_{L_1} < \varepsilon \).

Consider \( \varepsilon/5 > 0 \). By \( L_1 \)-total boundedness of the space \( \mathcal{M} \) there exist \( P_1, \ldots, P_k \) in \( \mathcal{M} \) such that for every \( P \in \mathcal{M} \), \( \inf \{ \| P - P_i \|_{L_1} : 1 \leq i \leq k \} < \varepsilon \). On the other hand

\[
\| P_i - P_j \|_{L_1} = P_i \left\{ x : \frac{dP_i}{d\mu}(x) > \frac{dP_j}{d\mu}(x) \right\} - P_j \left\{ x : \frac{dP_i}{d\mu}(x) > \frac{dP_j}{d\mu}(x) \right\},
\]

where the existence of the dominating measure \( \mu \) is secured by total boundedness. Consider all sets of the form \( \{ x : dP_i(x)/d\mu > dP_j(x)/d\mu \} \) for \( 1 \leq i < j \leq k \) and call them \( B_1, \ldots, B_n \).

Consider now \( V_{\varepsilon/5, 1, B_1, \ldots, B_n} = \{ (\tilde{P}, \tilde{Q}) : \sup_{1 \leq i \leq n} |\tilde{P}(B_i) - \tilde{Q}(B_i)| < \varepsilon/5 \} \). Let \( P, Q \in V_{\varepsilon/5, 1, B_1, \ldots, B_n} \). We will prove \( \| P - Q \|_{L_1} < \varepsilon \). Using the triangular inequality we get

\[
\| P - Q \|_{L_1} \leq \| P - P_i \|_{L_1} + \| P_i - P_j \|_{L_1} + \| P_j - Q \| < \frac{2\varepsilon}{5} + \| P_i - P_j \|_{L_1}.
\]

(\( A \) \( \Rightarrow \) \( B \)). To prove now that \( \mathcal{M} \) is \( L_1 \)-totally bounded, it suffices to prove that for every \( \varepsilon > 0 \) there exist \( P_1, \ldots, P_l \) in \( \mathcal{M} \) such that \( \inf \{ \| P - P_i \|_{L_1} : 1 \leq i \leq l \} < \varepsilon \) for every \( P \in \mathcal{M} \).

Fix \( \varepsilon > 0 \). Since \( \phi^* \) is uniformly continuous for that \( \varepsilon \) there exist \( \delta(\varepsilon) \), \( m(\varepsilon) \) and \( B_1, \ldots, B_n \) in \( \mathcal{A}^m(\varepsilon) \) such that if \( (P, Q) \in V_{\delta(\varepsilon), m(\varepsilon), B_1, \ldots, B_n} \) then \( \| \phi^*(P) - \phi^*(Q) \| = \| P - Q \|_{L_1} < \varepsilon \). By Lemma 3, \( U \) is precompact so for \( V_{\delta(\varepsilon), m(\varepsilon), B_1, \ldots, B_n} \) there exist \( P_1, \ldots, P_l \) such that \( \mathcal{M} = \bigcup_{k=1}^l P : \sup_{1 \leq i \leq n} |P^m(B_k) - P^m(B_k)| < \delta(\varepsilon) \). From this it follows that \( \inf \{ \| P - P_i \|_{L_1} : 1 \leq i \leq l \} < \varepsilon \) for every \( P \in \mathcal{M} \).

Lemma 5. Assume now that \( \mathcal{M} \) is \( L_1 \)-separable, \( \phi^* : (\mathcal{M}, U) \to (\phi^*(\mathcal{M}), \| \cdot \|_{L_1}) \), and \( \mu \) is a dominating measure existing by \( L_1 \)-separability assumption. The following propositions are equivalent:

(B) \( \mathcal{M} \) is \( L_1 \)-totally bounded.

Proof. (B) \( \Rightarrow \) (C). We will first prove (b). Fix \( \varepsilon > 0 \). For \( \varepsilon/2 \) there exist \( P_1, \ldots, P_n \) in \( \mathcal{M} \) such that \( \inf \{ \| P - P_i \|_{L_1} : 1 \leq i \leq n \} < \varepsilon/2 \) for every \( P \in \mathcal{M} \). By absolute continuity of \( P_1, \ldots, P_n \) with respect to \( \mu \) there is a \( \delta(\varepsilon) > 0 \) such that if \( \mu(A) < \delta(\varepsilon) \), then \( \sup \{ P(A) : 1 \leq i \leq n \} < \varepsilon/2 \). Consider now \( P \in \mathcal{M} \). Then

\[
P(A) = P(A) - P_i(A) + P_i(A) \leq \| P - P_i \|_{L_1} + P_i(A) < \varepsilon/2 + \varepsilon/2 = \varepsilon,
\]
where $P_\ell$ is the center of the $L_1$-ball of radius $\varepsilon/2$ containing $P$ and this holds for every $A \in \mathcal{A}$ such that $\mu(A) < \delta(\varepsilon)$ and for every $P$ in $\mathcal{M}$.

We now prove (a) by constructing a uniformly consistent minimum distance estimator.

Let $a_n$ be a sequence of numbers such that $a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots \geq 0$ tending to 0. Since $\mathcal{A}$ is $L_1$-totally bounded for every $a_n$, there is an $a_n$-dense subset of measures $P_1, \ldots, P_{N(a_n)}$ in $\mathcal{M}$. Let

$$\mathcal{F}_{a_n} = \left\{ x: \frac{dP_i}{d\mu}(x) > \frac{dP_j}{d\mu}(x), 1 \leq i < j \leq N(a_n) \right\}.$$ 

By applying the triangular inequality, we have then that for every $P, Q$ in $\mathcal{M}$,

$$\|P - Q\|_{L_1} \leq 4a_n + 2 \sup\{|P(A) - Q(A); A \in \mathcal{F}_{a_n}\}.$$

Let $X_1, \ldots, X_k$ be independent identically distributed observations from $P \in \mathcal{M}$ and $\mu_k(A) = \frac{1}{k} \sum_{i=1}^{k} I_A(X_i)$ the empirical measure indexed by $A \in \mathcal{A}$. Define an estimator

$$\hat{\phi}_k(n): \sup\{|\mu_k(A) - \hat{\phi}_k(n)(A); A \in \mathcal{F}_{a_n}\} = \inf\{\sup\{|\mu_k(A) - Q(A); A \in \mathcal{F}_{a_n}\}; Q \in \mathcal{M}\}$$

(without loss of generality we can assume the infimum is achieved). For fixed $a_n$, by the law of large numbers, $\sup\{|\mu_k(A) - \hat{\phi}_k(n)(A); A \in \mathcal{F}_{a_n}\} \leq \sup\{|\mu_k(A) - P(A); A \in \mathcal{F}_{a_n}\} \to 0$ as $k$ tends to infinity. So there is a $K(a_n)$ such that

$$\sup\{|\mu_k(A) - \hat{\phi}_k(n)(A); A \in \mathcal{F}_{a_n}\} \leq a_n \quad \text{for} \quad K(a_n).$$

Repeating the same construction for $\mathcal{F}_{a_{n+1}}$, there exists $K(a_{n+1})$ such that $\sup\{|\mu_k(A) - \hat{\phi}_k(n+1)(A); A \in \mathcal{F}_{a_{n+1}}\} < a_{n+1} \quad \text{for} \quad K(a_{n+1})$. Let $K(a_n, a_{n+1}) = \max\{K(a_n), K(a_{n+1})\}$ and define

$$\hat{\phi}_k = \begin{cases} \hat{\phi}_{k,n} & \text{for} \ K \leq K(a_n, a_{n+1}), \\ \hat{\phi}_{k,n+1} & \text{for} \ K(a_n, a_{n+1}) < K \leq K(a_{n+1}, a_{n+2}). \end{cases}$$

We claim that $\|\hat{\phi}_k - \phi\|_{L_1} \to 0$ as $k$ tends to infinity in $P^k$-probability. By construction,

$$\|\hat{\phi}_k - \phi\|_{L_1} \leq 4a_n + 2 \sup\{|\hat{\phi}_k(A) - P(A); A \in \mathcal{F}_{a_n}\} \leq 5a_n$$

for $K \geq K(a_n, a_{n+1})$ with high probability. Q.E.D.

$(C) \Rightarrow (B)$. By Lemma 4 it is enough to prove $(C) \Rightarrow (A)$, i.e. we will prove that $\phi^*$ is uniformly continuous.

By assumption there is a uniformly consistent estimator $T_n$ of $\phi^*(P)$ taking values in $\phi^*(\mathcal{M})$. So for every $n$, there is $\hat{\phi}_n$ in $\mathcal{M}$ such that $T_n = \phi^*(\hat{\phi}_n)$. By uniform consistency we will then have

$$(4) \quad \sup\{E_{P^n}\|T_n - \phi^*(P)\|; P \in \mathcal{M}\} = \sup\{E_{P^n}\|\hat{\phi}_n - P\|_{L_1}; P \in \mathcal{M}\} \to 0$$

as $n$ tends to infinity.
Fix $\epsilon > 0$ and consider
\[ ||\phi^*(P) - \phi^*(Q)|| = ||P - Q||_{L_1} \leq ||P - E_{P^*}\hat{P}_n||_{L_1} + ||E_{P^*}\hat{P}_n - E_{Q^*}\hat{P}_n||_{L_1} + ||E_{Q^*}\hat{P}_n - Q||_{L_1} \leq E_{P^*}||P - \hat{P}_n||_{L_1} + ||E_{P^*}\hat{P}_n - E_{Q^*}\hat{P}_n||_{L_1} + ||E_{Q^*}\hat{P}_n - Q||_{L_1} \leq 2\epsilon + ||E_{P^*}\hat{P}_n - E_{Q^*}\hat{P}_n||_{L_1} \]
for all $n \geq n(\epsilon)$ by (4). Let $n = n(\epsilon)$ for the rest of the proof.

Observe now that $\hat{P}_n$ takes values in the space $M$ which is $L_1$-separable. By separability there exists a dominating measure $\mu$. Also strong measurability is equivalent to weak measurability (so we do not have any measurability problems) and there exist sets $A^n_i$ in $A^n$, $i = 1, 2, \ldots$, such that $||\hat{P}_n - \sum_{i=1}^{\infty} I_{A^n_i} P||_{L_1} < \epsilon$ a.e. $\mu^n$, with $\{P_i\}$, $1 \leq i \leq n$, being the countable dense subset of $M$.

So we finally have
\[
(5) \quad ||\phi^*(P) - \phi^*(Q)|| \leq 4\epsilon + E_{P^n} \sum_{i=1}^{\infty} I_{A^n_i} P_i - E_{Q^n} \sum_{i=1}^{\infty} I_{A^n_i} P_i \]
As we know, $\sum_{i=1}^{\infty} I_{A^n_i} P_i = \lim_{k \to \infty} \sum_{i=1}^{k} I_{A^n_i} P_i$ in $L_1$, $\mu^n$ a.s. and the convergence is almost uniform $\mu^n$, so for $\delta(\epsilon) > 0$ there exists $E_\delta$ in $A^n$ such that $\mu^n(E_\delta) < \delta(\epsilon)$ and for $K = K(\epsilon)$ big enough $||\sum_{i=1}^{\infty} I_{A^n_i} P_i - \sum_{i=1}^{k} I_{A^n_i} P_i||_{L_1} < \epsilon$ for all $x$ in $X^n - E_\epsilon$.

We will try now to evaluate the right-hand side of (5):
\[
E_{P^n} \sum_{i=1}^{\infty} I_{A^n_i} P_i - E_{Q^n} \sum_{i=1}^{\infty} I_{A^n_i} P_i \]
\[ = \left| \int_{E_{\delta}} \left( \sum_{i=1}^{\infty} I_{A^n_i} P_i - \sum_{i=1}^{k} I_{A^n_i} P_i \right) dP^n + \int_{X^n - E_{\delta}} \left( \sum_{i=1}^{\infty} I_{A^n_i} P_i - \sum_{i=1}^{k} I_{A^n_i} P_i \right) dP^n \right|_{L_1} \leq 2P^n(E_\epsilon) + \epsilon \cdot P^n(\mathcal{X}^n - E_\epsilon) \leq 3\epsilon.
\]
Repeating the same type of calculation for $E_{Q^n} \sum_{i=1}^{\infty} I_{A^n_i} P_i$ and replacing in (5), we get
\[
||\phi^*(P) - \phi^*(Q)|| \leq 10\epsilon + E_{P^n} \sum_{i=1}^{k} I_{A^n_i} P_i - E_{Q^n} \sum_{i=1}^{k} I_{A^n_i} P_i \]
\[ = 10\epsilon + \sup \left| \left( \sum_{i=1}^{k} (P^n(A^n_i) - Q^n(A^n_i)) \cdot P_i(A) \right) ; A \in \mathcal{A} \right| \leq 10\epsilon + K \cdot \sup \left| \left( P^n(A^n_i) - Q^n(A^n_i) \right) ; 1 \leq i \leq k \right| . \quad \text{Q.E.D.}
\]
From Lemmas 4 and 5 we have the following

**Theorem 2.** Let $M$ be an $L_1$-separable family of measures, $\phi^*: (M, U) \to (\phi^*(M), \| \cdot \|_{L_1})$ and $\mu$ a dominating measure. The following positions are equivalent.
(A) $\phi^*$ is uniformly continuous.
(B) $M$ is $L_1$-totally bounded.
(C) (a) There exists a uniformly consistent estimator for \( \phi^*(P) \) with values in \( \phi^*(\mathcal{M}) \), (b) for every \( \varepsilon > 0 \) there exists a \( \delta(\varepsilon) > 0 \) such that for every \( A \) in \( \mathcal{A} \):
\[
\mu(A) < \delta(\varepsilon), \sup\{ P(A); P \in \mathcal{M} \} < \varepsilon.
\]

**Remark.** This theorem shows that in the case that \( \mathcal{M} \) is uniformly dominated every uniformly consistent estimator in \( L_1 \) can be achieved through minimum distance. On the other hand the condition of uniform domination is not a necessary condition for the existence of uniformly consistent estimates in \( L_1 \) as the example of families of densities satisfying the Hoeffding-Wolfowitz [1958] condition shows.

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