ON THE EXISTENCE OF UNIFORMLY CONSISTENT ESTIMATES

Yannis G. Yatracos

Abstract. Let \( \mathcal{M} \) be a family of probability measures on \((X, \mathcal{A})\) and \( U \) the uniform structure defined by vicinities of the form

\[
\left\{ (P, Q) : \sup_{1 \leq i \leq K} |P^n(A_{i,n}) - Q^n(A_{i,n})| < \epsilon \right\},
\]

where \( P^n \) is the product measure on \((X^n, \mathcal{A}^n)\), \( A_{i,n} \in \mathcal{A}^n, \epsilon > 0, n \wedge K \geq 1 \). Let \( \phi^*(\mathcal{M}, U) \to (\phi^*(\mathcal{M}), d) \), where

\[
d(\phi^*(P), \phi^*(Q)) = \|P - Q\|_{L_1} = 2 \sup_{A \in \mathcal{A}} |P(A) - Q(A)|.
\]

We consider the case where the space of measures \( M \) is \( L_1 \) separable and relate the existence of uniformly consistent estimates for \( \phi^*(P) \) with uniform continuity of \( \phi^* \) and \( L_1 \)-total boundedness of \( M \).

1. Introduction, notation and definitions. Let \((X, \mathcal{A})\) be a space with a \( \sigma \)-field, let \( \mathcal{M} \) be a family of probability measures on \( \mathcal{A} \), \((X^n, \mathcal{A}^n)\) the \( n \)th product space and \( \sigma \)-field, and let \( X_1, \ldots, X_n \) be independent identically distributed observations according to some measure \( P \in \mathcal{M} \), \( P^n \) being the \( n \)th product measure. Let \( \Theta \) be a topological space which is homeomorphic to a subset of \([0,1]^\infty\), \( h \) being the homeomorphism, and let \( \rho \) be a metric on \([0,1]^\infty\) of the form

\[
\rho(x, y) = \sum_{m=1}^{\infty} 2^{-m}|x_m - y_m|,
\]

where \( x_m, y_m \) are coordinates of \( x, y \) respectively. Let \( \phi^*: P \to \phi^*(P) \) be a function defined on \( \mathcal{M} \) with values in \( \Theta \), and \( \phi = h \circ \phi^*, \phi: \mathcal{M} \to ([0,1]^\infty, \rho) \).

LeCam and Schwartz [1960] gave necessary and sufficient conditions for the existence of uniformly consistent estimates of \( \phi(P) \) in terms of the uniform continuity of \( \phi \) with respect to a uniform structure \( U = \bigcup_n U_n \), where each \( U_n \) consists of vicinities of the form \((P, Q) : \sup_{1 \leq i \leq K} |f_i dP^n - f_i dQ^n| < 1 \) for \( f_1, \ldots, f_l \) bounded measurable functions on the product space \((X^n, \mathcal{A}^n)\). The same uniform structure has been used by Pfanzagl [1968] and Moché [1977]. Under the above set-up we will explain why it is natural to consider the uniform structure \( U \) and we
will offer a theorem on the existence of estimates. Relaxing the condition of homeomorphism of $\Theta$ with a subset of $[0,1]^\infty$ and assuming only that $(\phi^*(\mathcal{M}), d)$ is separable when metrized with the total variation $(L_1)$ norm between the measures, i.e. $d(\phi^*(P), \phi^*(Q)) = \|P - Q\|_{L_1} = 2\sup\{|P(A) - Q(A)|; A \in \mathcal{A}\}$, we offer a theorem of the same type in the form of equivalent propositions.

**Definition.** We shall say $\phi$ is uniformly consistently estimable if there is a sequence $T_n$ of measurable functions from $(\mathcal{A}^n, \mathcal{A}^n)$ to $([0,1]^\infty, \rho)$ such that for every $\epsilon > 0$, $\sup\{\rho(T_n, \phi(P)) > \epsilon; P \in \mathcal{M}\} \to 0$ as $n$ tends to infinity.

**Remark.** One can easily see that since $\rho$ is bounded, the above definition is equivalent to $\sup\{E_p \rho(T_n, \phi(P)); P \in \mathcal{M}\} \to 0$ as $n$ tends to infinity.

Recall from topology the following

**Definition.** Let $(Y, d)$ be a metric space with a metric $d$. We say $Y$ is $d$-totally bounded if for every $\epsilon > 0$ there are $y_1, \ldots, y_n(\epsilon)$ elements of $Y$ such that

$$Y = \bigcup_{i=1}^{n(\epsilon)} \{y: d(y, y_i) < \epsilon\}.$$  

In our effort to relax the hypothesis of homeomorphism of $\Theta$ with $[0,1]^\infty$ our estimator will take values in an abstract space. Terminology and results under this set-up concerning measurability (strong and weak), convergence (almost sure and almost uniform) and Bochner integration can be found in Hille and Phillips [1957, §§3.5 and 3.7]. For the notion of uniform structure the reader is referred to Choquet [1969, Chapter 2, §5]. For a deep insight on consistency questions, we suggest the remarkable paper of Kraft [1955].


**Lemma 1.** Assume $\phi$ is uniformly consistently estimable by $T_n$. Then for every $\epsilon > 0$ there exist positive integers $n = n(\epsilon), K = K(\epsilon)$ and $l = l(\epsilon)$, and measurable sets $B_1^n, \ldots, B_l^n$ of the $n$th product $\sigma$-field $\mathcal{A}^n$ such that

$$\rho(\phi(P), \phi(Q)) \leq \epsilon + K \cdot \sup\{|P^n(B_j^n) - Q^n(B_j^n)|; 1 \leq j \leq l(\epsilon)\}.$$  

**Proof.** We will use the triangular inequality and then Jensen’s inequality,

$$\rho(\phi(P), \phi(Q)) \leq \rho(\phi(P), E_{P^n}T_n) + \rho(E_{P^n}T_n, E_{Q^n}T_n) + \rho(E_{Q^n}T_n, \phi(Q))$$

$$\leq E_{P^n} \rho(\phi(P), T_n) + \rho(E_{P^n}T_n, E_{Q^n}T_n) + E_{Q^n} \rho(T_n, \phi(Q))$$

$$\leq 2 \cdot \frac{\epsilon}{4} + \rho(E_{P^n}T_n, E_{Q^n}T_n)$$

for every $n \geq n(\epsilon)$ by uniform convergence of $T_n$ to $\phi(P)$. For the rest of the proof let $n = n(\epsilon)$.

Consider

$$\rho(E_{P^n}T_n, E_{Q^n}T_n) = \sum_{m=1}^{\infty} 2^{-m}|E_{P^n}(T_n)_m - E_{Q^n}(T_n)_m|,$$
where \((T_n)_m\) are the coordinates of \(T_n\), \(1 \leq m < \infty\). By assumption \(\|E_p^n(T_n)_m - E_{Q^n}(T_n)_m\| \leq 2\), so there is \(m_0 = m_0(\varepsilon) > 0\) such that

\[
\sum_{m = m_0 + 1}^{\infty} 2^{-m}\|E_p^n(T_n)_m - E_{Q^n}(T_n)_m\| \leq \sum_{m = m_0 + 1}^{\infty} 2^{-m} \leq \frac{\varepsilon}{4}.
\]

So now (2) becomes

\[
(3) \quad \rho(E_p^nT_n, E_{Q^n}T_n) \leq \frac{\varepsilon}{4} + \sum_{m=1}^{m_0} 2^{-m}\|E_p^n(T_n)_m - E_{Q^n}(T_n)_m\|.
\]

Consider now \((T_n)_m\) for some \(m \in \{1, \ldots, m_0\}\). Since \(0 \leq (T_n)_m \leq 1\) for \(\varepsilon/8\) there exists a simple function of the form \(\sum_{i=1}^{l_m(\varepsilon)} a_{i, m} I_{A^n_{i, m}}\) such that

\[
\sup_{x \in \mathcal{X}} \left( \left\| (T_n(x_1, \ldots, x_n))_m - \sum_{i=1}^{l_m(\varepsilon)} a_{i, m} I_{A^n_{i, m}} \right\| ; (x_1, \ldots, x_n) \in \mathcal{X}^n \right) \leq \frac{\varepsilon}{8}.
\]

The \(m\)th term of the sum in (3) becomes

\[
\|E_p^n(T_n)_m - E_{Q^n}(T_n)_m\| \leq \frac{2\varepsilon}{8} + \sum_{i=1}^{l_m(\varepsilon)} a_{i, m}\|P^n(A^n_{i, m}) - Q^n(A^n_{i, m})\|
\]

\[
\leq \frac{\varepsilon}{4} + \left( \sum_{i=1}^{l_m(\varepsilon)} |a_{i, m}| \right) \cdot \sup \left\{ |P^n(A^n_{i, m}) - Q^n(A^n_{i, m})| ; 1 \leq i \leq l_m(\varepsilon) \right\}.
\]

Letting \(K_m(\varepsilon) = \sum_{i=1}^{l_m(\varepsilon)} |a_{i, m}|\) and repeating the same argument for all \(m \in \{1, \ldots, m_0\}\) in (3) we get

\[
\rho(E_p^nT_n, E_{Q^n}T_n) \leq \frac{\varepsilon}{4} + \sum_{m=1}^{m_0} 2^{-m} \cdot \frac{\varepsilon}{4}
\]

\[
+ \sum_{m=1}^{m_0} 2^{-m} \cdot K_m(\varepsilon) \cdot \sup \left\{ |P^n(A^n_{i, m}) - Q^n(A^n_{i, m})| ; 1 \leq i \leq l_m(\varepsilon) \right\}
\]

\[
\leq \frac{\varepsilon}{2} + \left( \sum_{m=1}^{m_0} 2^{-m} K_m(\varepsilon) \right) \cdot \sup \left\{ |P^n(B^n_j) - Q^n(B^n_j)| ; 1 \leq j \leq \sum_{m=1}^{m_0} l_m(\varepsilon) \right\}.
\]

By letting \(K(\varepsilon) = \sum_{m=1}^{m_0} 2^{-m} K_m(\varepsilon), l(\varepsilon) = \sum_{m=1}^{m_0} l_m(\varepsilon)\) and replacing in (1) we get

\[
\rho(\phi(P), \phi(Q)) \leq \varepsilon + K \cdot \sup \left\{ |P^n(B^n_j) - Q^n(B^n_j)| ; 1 \leq j \leq l \right\}.
\]

Q.E.D.

The above lemma shows that giving \(\mathcal{M}\) the uniform structure consisting of vicinities of the form \(\{(P, Q) : \sup \{ |P^n(A^n_i) - Q^n(A^n_i)| ; 1 \leq i \leq l \} < \varepsilon \}\) for \(l \in N^+, n \in N^+, \varepsilon > 0\), is the natural way for associating uniform convergence in probability of \(T_n\) with the uniform continuity of \(\phi: \mathcal{M} \to ([0, 1]^\infty, \rho)\).

It is now easy to see that these vicinities give rise to the same uniform structure \(U\) as those in the introduction.
We offer the following

**Lemma 2.** Let \( \phi : (\mathcal{M}, U) \to ([0,1]^\infty, \rho) \). The following are equivalent:

1. \( \phi \) is uniformly continuous.
2. For every \( \varepsilon > 0 \) there exists \( K(\varepsilon), m(\varepsilon), n(\varepsilon) \) all positive, and \( B_1^{m(\varepsilon)}, \ldots, B_n^{m(\varepsilon)} \) elements of the \( m(\varepsilon) \)-product \( \sigma \)-field \( \mathcal{M}^{m(\varepsilon)} \) such that

\[
\rho(\phi(P), \phi(Q)) \leq \varepsilon + K(\varepsilon) \cdot \sup\{|P^{m(\varepsilon)}(B_i^{m(\varepsilon)}) - Q^{m(\varepsilon)}(B_i^{m(\varepsilon)})|; 1 \leq i \leq n(\varepsilon)\}.
\]

**Proof.** (2) \( \Rightarrow \) (1). Obvious.

(1) \( \Rightarrow \) (2). Since \( \phi \) is uniformly continuous, for every \( \varepsilon > 0 \) there is \( \delta(\varepsilon) > 0 \), \( m = m(\varepsilon) > 0 \), \( n = n(\varepsilon) > 0 \) and sets \( B_1^{m(\varepsilon)}, \ldots, B_n^{m(\varepsilon)} \) in \( \mathcal{M}^{m(\varepsilon)} \) such that if

\[
\sup\{|P^{m(\varepsilon)}(E) - Q^{m(\varepsilon)}(E)|; 1 \leq i \leq n(\varepsilon)\} < \delta(\varepsilon),
\]

then \( \rho(\phi(P), \phi(Q)) < \varepsilon \).

From this, for every \( P, Q \) such that \( \rho(\phi(P), \phi(Q)) \geq \varepsilon \) it is necessary that

\[
\sup\{|P^{m}(B_i^{m}) - Q^{m}(B_i^{m})|; 1 \leq i \leq n \} \geq \delta(\varepsilon).
\]

Going to the statement it is required to prove: Note that for every \( \varepsilon > 0 \) either

\[
\rho(\phi(P), \phi(Q)) < \varepsilon \text{ or } \rho(\phi(P), \phi(Q)) \geq \varepsilon.
\]

For the second case we have then

\[
\rho(\phi(P), \phi(Q)) \leq \rho(\phi(P), \phi(Q)) \cdot \left| \inf\left\{ \sup\{|P^{m}(B_i^{m}) - Q^{m}(B_i^{m})|; 1 \leq i \leq n \} \right\} \right|^{-1}\cdot \sup\{|P^{m}(B_i^{m}) - Q^{m}(B_i^{m})|; 1 \leq i \leq n \}
\]

\[
\leq 2[\delta(\varepsilon)]^{-1}\sup\{|P^{m}(B_i^{m}) - Q^{m}(B_i^{m})|; 1 \leq i \leq n \}.
\]

Let \( 2[\delta(\varepsilon)]^{-1} = K(\varepsilon) \). So finally for all \( P, Q \)

\[
\rho(\phi(P), \phi(Q)) \leq \varepsilon + K(\varepsilon) \cdot \sup\{|P^{m}(B_i^{m}) - Q^{m}(B_i^{m})|; 1 \leq i \leq n \}.
\]

We offer

**Theorem 1.** Let \( \phi : (\mathcal{M}, U) \to ([0,1]^\infty, \rho) \) with \( U \) defined as above. The following statements are equivalent:

1. \( \phi \) is uniformly continuous.
2. There exists a uniformly consistent estimator of \( \phi(P) \).

**Proof.** (1) \( \Rightarrow \) (2). Given in LeCam and Schwartz [1960, Theorem 1].

(2) \( \Rightarrow \) (1). Corollary of Lemmas 1 and 2.

3. **The main result.** Consider now the case \( \phi^* : (\mathcal{M}, U) \to (\phi^*(\mathcal{M}), \| \cdot \|_{L_1}) \), as in the introduction, with \( \phi^*(\mathcal{M}) \) separable when metrized with the \( L_1 \)-distance between measures. We will give our result in a series of lemmas.

**Lemma 3.** The uniform structure \( U \) is precompact (in the sense that for any vicinity of the form \{ \( (P, Q): \sup\{|P^{m}(A_i^{m}) - Q^{m}(A_i^{m})|; 1 \leq i \leq n \} < \varepsilon \} \) there exist \( P_1, \ldots, P_l \) in \( \mathcal{M} \) such that \( \mathcal{M} = \bigcup_{k=1}^{l} \{ (P: \sup\{|P^{m}(A_i^{m}) - P_k^{m}(A_i^{m})|; 1 \leq i \leq n \} < \varepsilon \} \).

**Proof.** Given in LeCam and Schwartz [1960, p. 142].
Lemma 4. Let \( \phi^* : (\mathcal{M}, U) \to (\phi^*(\mathcal{M}), \| \cdot \|_{L_1}) \). The following propositions are equivalent:

(A) \( \phi^* \) is uniformly continuous.

(B) \( \mathcal{M} \) is \( L_1 \)-totally bounded.

Proof. (B) \( \Rightarrow \) (A). To prove \( \phi^* \) is uniformly continuous, it suffices to prove that for every \( \epsilon > 0 \) there exist \( \delta(\epsilon) \) and \( B_1, \ldots, B_n \) sets in \( \mathcal{A}^m \) such that for every

\[
(P, Q) \in V_{\delta(\epsilon), m, B_1, \ldots, B_n} = \{ (\hat{P}, \hat{Q}) : \sup |\hat{P}^m(B_i) - \hat{Q}^m(B_i)| ; 1 \leq i \leq n \} < \delta(\epsilon)
\]

implies \( \| \phi^*(P) - \phi^*(Q) \| = \| P - Q \|_{L_1} < \epsilon \).

Consider \( \epsilon/5 > 0 \). By \( L_1 \)-total boundedness of the space \( \mathcal{M} \) there exist \( P_1, \ldots, P_k \) in \( \mathcal{M} \) such that for every \( P \in \mathcal{M} \), \( \inf \{ \| P - P_i \|_{L_1} ; 1 \leq i \leq k \} \leq \epsilon \). On the other hand

\[
\| P_i - P_j \|_{L_1} = P_i \left( x : \frac{dP_i}{d\mu}(x) > \frac{dP_j}{d\mu}(x) \right) - P_j \left( x : \frac{dP_i}{d\mu}(x) > \frac{dP_j}{d\mu}(x) \right).
\]

where the existence of the dominating measure \( \mu \) is secured by total boundedness. Consider all sets of the form \( \{ x : dP_i(x)/d\mu > dP_j(x)/d\mu \} \) for \( 1 \leq i < j \leq k \) and call them \( B_1, \ldots, B_n \).

Consider now \( V_{\epsilon/5, 1, B_1, \ldots, B_n} = \{ (\hat{P}, \hat{Q}) : \sup |\hat{P}(B_i) - \hat{Q}(B_i)| ; 1 \leq i \leq n \} < \epsilon/5 \} \).

Let \( P, Q \in V_{\epsilon/5, 1, B_1, \ldots, B_n} \). We will prove \( \| P - Q \|_{L_1} < \epsilon \). Using the triangular inequality we get

\[
\| P - Q \|_{L_1} \leq \| P - P_i \|_{L_1} + \| P_i - P_j \|_{L_1} + \| P_j - Q \| < \frac{2\epsilon}{5} + \| P_i - P_j \|_{L_1} \leq \frac{4\epsilon}{5} + \sup \{ |P(B_i) - Q(B_i)| ; 1 \leq i \leq n \} < \epsilon.
\]

(A) \( \Rightarrow \) (B). To prove now that \( \mathcal{M} \) is \( L_1 \)-totally bounded, it suffices to prove that for every \( \epsilon > 0 \) there exist \( P_1, \ldots, P_l \) in \( \mathcal{M} \) such that \( \inf \{ \| P - P_i \|_{L_1} ; 1 \leq i \leq l \} \leq \epsilon \) for every \( P \in \mathcal{M} \).

Fix \( \epsilon > 0 \). Since \( \phi^* \) is uniformly continuous for that \( \epsilon \) there exist \( \delta(\epsilon) \), \( m(\epsilon) \) and \( B_1, \ldots, B_n \) in \( \mathcal{A}^{m(\epsilon)} \) such that if \( (P, Q) \in V_{\delta(\epsilon), m, B_1, \ldots, B_n} \) then \( \| \phi^*(P) - \phi^*(Q) \| = \| P - Q \|_{L_1} < \epsilon \). By Lemma 3, \( U \) is precompact so for \( V_{\delta(\epsilon), m, B_1, \ldots, B_n} \) there exist \( P_1, \ldots, P_l \) such that \( \mathcal{M} = \bigcup_{k=1}^l \{ P : \sup \{ |P^{m(\epsilon)}(B_j) - P^{m(\epsilon)}(B_j)| ; 1 \leq i \leq n \} < \delta(\epsilon) \} \).

From this it follows that \( \inf \{ \| P - P_i \|_{L_1} ; 1 \leq i \leq l \} \leq \epsilon \) for every \( P \in \mathcal{M} \).

Lemma 5. Assume now that \( \mathcal{M} \) is \( L_1 \)-separable, \( \phi^* : (\mathcal{M}, U) \to (\phi^*(\mathcal{M}), \| \cdot \|_{L_1}) \), and \( \mu \) is a dominating measure existing by \( L_1 \)-separability assumption. The following propositions are equivalent:

(B) \( \mathcal{M} \) is \( L_1 \)-totally bounded.

(C) (a) There exists a uniformly consistent estimator for \( \phi^*(P) \) with values in \( \phi^*(\mathcal{M}) \), (b) for every \( \epsilon > 0 \) there exists a \( \delta(\epsilon) > 0 \) such that if \( \mu(A) < \delta(\epsilon) \), then \( \sup \{ P(A) ; P \in \mathcal{M} \} < \epsilon \).

Proof. (B) \( \Rightarrow \) (C). We will first prove (b). Fix \( \epsilon > 0 \). For \( \epsilon/2 \) there exist \( P_1, \ldots, P_n \) in \( \mathcal{M} \) such that \( \inf \{ \| P - P_i \|_{L_1} ; 1 \leq i \leq n \} \leq \epsilon/2 \) for every \( P \in \mathcal{M} \). By absolute continuity of \( P_1, \ldots, P_n \) with respect to \( \mu \) there is a \( \delta(\epsilon) > 0 \) such that if \( \mu(A) < \delta(\epsilon) \), then

\[
\sup \{ P_i(A) ; 1 \leq i \leq n \} < \epsilon/2.
\]

Consider now \( P \in \mathcal{M} \). Then

\[
P(A) = P(A) - P_i(A) + P_i(A) \leq \| P - P_i \|_{L_1} + P_i(A) < \epsilon/2 + \epsilon/2 = \epsilon,
\]
where \( P \) is the center of the \( L_1 \)-ball of radius \( \varepsilon/2 \) containing \( P \) and this holds for every \( A \in \mathcal{A} \) such that \( \mu(A) < \delta(\varepsilon) \) and for every \( P \) in \( \mathcal{M} \).

We now prove (a) by constructing a uniformly consistent minimum distance estimator.

Let \( a_n \) be a sequence of numbers such that \( a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots \geq 0 \) tending to 0. Since \( \mathcal{M} \) is \( L_1 \)-totally bounded for every \( a_n \), there is an \( a_n \)-dense subset of measures \( P_1, \ldots, P_N(a_n) \) in \( \mathcal{M} \). Let

\[
\mathcal{F}_{a_n} = \left\{ \{ x : \frac{dP_i}{d\mu}(x) > \frac{dP_j}{d\mu}(x) \} , 1 \leq i < j \leq N(a_n) \right\}.
\]

By applying the triangular inequality, we have then that for every \( P, Q \) in \( \mathcal{M} \),

\[
\| P - Q \|_{L_1} \leq 4a_n + 2\sup\{ |P(A) - Q(A)| ; A \in \mathcal{F}_{a_n} \}.
\]

Let \( X_1, \ldots, X_k \) be independent identically distributed observations from \( P \in \mathcal{M} \) and \( \mu_k(A) = \frac{1}{k} \sum_{i=1}^{k} I_A(X_i) \) the empirical measure indexed by \( A \in \mathcal{A} \). Define an estimator

\[
\hat{P}_{k,n} = \inf\{ \sup\{ |\mu_k(A) - Q(A)| ; A \in \mathcal{F}_{a_n} \} ; Q \in \mathcal{M} \}
\]

(without loss of generality we can assume the infimum is achieved). For fixed \( a_n \), by the law of large numbers, \( \sup\{ |\mu_k(A) - \hat{P}_{k,n}(A)| ; A \in \mathcal{F}_{a_n} \} \leq \sup\{ |\mu_k(A) - P(A)| ; A \in \mathcal{F}_{a_n} \} \to 0 \) as \( k \) tends to infinity. So there is a \( K(a_n) \) such that

\[
\sup\{ |\mu_k(A) - \hat{P}_{k,n}(A)| ; A \in \mathcal{F}_{a_n} \} \leq a_n \quad \text{for} \quad K \geq K(a_n).
\]

Repeating the same construction for \( \mathcal{F}_{a_{n+1}} \), there exists \( K(a_{n+1}) \) such that

\[
\sup\{ |\mu_k(A) - \hat{P}_{k,n+1}(A)| ; A \in \mathcal{F}_{a_{n+1}} \} \leq a_{n+1} \quad \text{for} \quad K \geq K(a_{n+1}).
\]

Let \( K(a_n, a_{n+1}) = \max\{ K(a_n), K(a_{n+1}) \} \) and define

\[
\hat{P}_k = \begin{cases} 
\hat{P}_{k,n} & \text{for} \quad K \leq K(a_n, a_{n+1}), \\
\hat{P}_{k,n+1} & \text{for} \quad K(a_n, a_{n+1}) < K \leq K(a_{n+1}, a_{n+2}).
\end{cases}
\]

We claim that \( \| \hat{P}_k - P \|_{L_1} \to 0 \) as \( k \) tends to infinity in \( P \)-probability. By construction,

\[
\| \hat{P}_k - P \|_{L_1} \leq 4a_n + 2\sup\{ |\hat{P}_k(A) - P(A)| ; A \in \mathcal{F}_{a_n} \} \leq 5a_n
\]

for \( K \geq K(a_n, a_{n+1}) \) with high probability. Q.E.D.

(C) \( \Rightarrow \) (B). By Lemma 4 it is enough to prove (C) \( \Rightarrow \) (A), i.e. we will prove that \( \phi^* \) is uniformly continuous.

By assumption there is a uniformly consistent estimator \( T_n \) of \( \phi^*(P) \) taking values in \( \phi^*(\mathcal{M}) \). So for every \( n \), there is \( \hat{P}_n \) in \( \mathcal{M} \) such that \( T_n = \phi^*(\hat{P}_n) \). By uniform consistency we will then have

\[
\sup\{ E_{P_n}\| T_n - \phi^*(P) \| ; P \in \mathcal{M} \} = \sup\{ E_{P_n}\| \hat{P}_n - P \|_{L_1} ; P \in \mathcal{M} \} \to 0
\]
as \( n \) tends to infinity.
Fix $\epsilon > 0$ and consider

$$
\|\phi^*(P) - \phi^*(Q)\| = \|P - Q\|_{L_1}
\leq \|P - E_{\epsilon_1} \hat{P}_n\|_{L_1} + \|E_{\epsilon_1} \hat{P}_n - E_{\epsilon_1} \hat{Q}_n\|_{L_1} + \|E_{\epsilon_1} \hat{Q}_n - Q\|_{L_1}
\leq E_{\epsilon_1} \|P - \hat{P}_n\|_{L_1} + \|E_{\epsilon_1} \hat{P}_n - E_{\epsilon_1} \hat{Q}_n\|_{L_1} + \|E_{\epsilon_1} \hat{Q}_n - Q\|_{L_1}
\leq 2\epsilon + \|E_{\epsilon_1} \hat{P}_n - E_{\epsilon_1} \hat{Q}_n\|_{L_1}
$$

for all $n \geq n(\epsilon)$ by (4). Let $n = n(\epsilon)$ for the rest of the proof.

Observe now that $\hat{P}_n$ takes values in the space $\mathcal{M}$ which is $L_1$-separable. By separability there exists a dominating measure $\mu$. Also strong measurability is equivalent to weak measurability (so we do not have any measurability problems) and there exist sets $A^n_i$ in $\mathcal{A}$, $i = 1, 2, \ldots$, such that $\|\hat{P}_n - \sum_{i=1}^\infty I_{A^n_i} P_i\|_{L_1} < \epsilon$ a.e. $\mu^n$, with $\{ P_i \}, 1 \leq i \leq n$, being the countable dense subset of $\mathcal{M}$.

So we finally have

$$
(5) \quad \|\phi^*(P) - \phi^*(Q)\| \leq 4\epsilon + \|E_{\epsilon_1} \sum_{i=1}^\infty I_{A^n_i} P_i - E_{\epsilon_1} \sum_{i=1}^\infty I_{A^n_i} P_i\|.
$$

As we know, $\sum_{i=1}^\infty I_{A^n_i} P_i = \lim_{k \to \infty} \sum_{i=1}^k I_{A^n_i} P_i$ in $L_1$, $\mu^n$ a.s. and the convergence is almost uniform $\mu^n$, so for $\delta(\epsilon) > 0$ there exists $E_\epsilon$ in $\mathcal{A}$ such that $\mu^n(E_\epsilon) < \delta(\epsilon)$ and for $K = K(\epsilon)$ big enough $\|\sum_{i=1}^\infty I_{A^n_i} P_i - \sum_{i=1}^k I_{A^n_i} P_i\|_{L_1} < \epsilon$ for all $x$ in $\mathcal{X}^n - E_\epsilon$.

We will try now to evaluate the right-hand side of (5):

$$
\|E_{\epsilon_1} \sum_{i=1}^\infty I_{A^n_i} P_i - E_{\epsilon_1} \sum_{i=1}^k I_{A^n_i} P_i\|
= \left\| \int_{\mathcal{X}^n - E_\epsilon} \left( \sum_{i=1}^\infty I_{A^n_i} P_i - \sum_{i=1}^k I_{A^n_i} P_i \right) dP^n + \int_{E_\epsilon} \left( \sum_{i=1}^\infty I_{A^n_i} P_i - \sum_{i=1}^k I_{A^n_i} P_i \right) dP^n \right\|_{L_1}
\leq 2P^n(E_\epsilon) + \epsilon \cdot P^n(\mathcal{X}^n - E_\epsilon) \leq 3\epsilon.
$$

Repeating the same type of calculation for $E_{\epsilon_1} \sum_{i=1}^\infty I_{A^n_i} P_i$ and replacing in (5), we get

$$
\|\phi^*(P) - \phi^*(Q)\| \leq 10\epsilon + \|E_{\epsilon_1} \sum_{i=1}^k I_{A^n_i} P_i - E_{\epsilon_1} \sum_{i=1}^k I_{A^n_i} P_i\|
= 10\epsilon + \sup \left\{ \left| \sum_{i=1}^k (P^n(A^n_i) - Q^n(A^n_i)) \cdot P_i(A) \right| ; A \in \mathcal{A} \right\}
\leq 10\epsilon + K \cdot \sup \left\{ \left| P^n(A^n_i) - Q^n(A^n_i) \right| ; 1 \leq i \leq k \right\}.
$$

From Lemmas 4 and 5 we have the following

**Theorem 2.** Let $\mathcal{M}$ be an $L_1$-separable family of measures, $\phi^*: (\mathcal{M}, U) \to (\phi^*(\mathcal{M}), \| \cdot \|_{L_1})$ and $\mu$ a dominating measure. The following positions are equivalent.

(A) $\phi^*$ is uniformly continuous.

(B) $\mathcal{M}$ is $L_1$-totally bounded.
(C) (a) There exists a uniformly consistent estimator for $\phi^*(P)$ with values in $\phi^*(\mathcal{M})$, (b) for every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that for every $A$ in $\mathcal{A}$: $\mu(A) < \delta(\varepsilon)$, $\sup\{P(A); P \in \mathcal{M}\} < \varepsilon$.

Remark. This theorem shows that in the case that $\mathcal{M}$ is uniformly dominated every uniformly consistent estimator in $L_1$ can be achieved through minimum distance. On the other hand the condition of uniform domination is not a necessary condition for the existence of uniformly consistent estimates in $L_1$ as the example of families of densities satisfying the Hoeffding-Wolfowitz [1958] condition shows.

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Department of Statistics, University of California, Berkeley, California 94720

Current address: Department of Statistics, Hill Center for the Mathematical Sciences, Busch Campus, Rutgers University, New Brunswick, New Jersey 08903