PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 94, Number 3, July 1985

A LATTICE OF CONDITIONS ON TOPOLOGICAL SPACES

P. J. COLLINS, G. M. REED,[1] A. W. ROSCOE AND M. E. RUDIN

ABSTRACT. If $W(x)$ (for each $x \in X$) is a family of subsets each containing x, various conditions on $\{W(x): x \in X\}$ are investigated. They yield new criteria for paracompactness, metrisability and the existence of a semimetric generating a given topology.

1. Introduction. The aim of this paper is to answer some questions raised by two of the authors in [3] and to shed further light on the type of conditions discussed there. The conditions we shall be investigating are all variants on (F) and (G) of the earlier paper, which we now specify. They apply to topological spaces X for each element x of which a family $W(x)$ of subsets containing x is given. Let $\mathscr{W} = \{W(x): x \in X\}$. We say that $\mathscr{W}$ satisfies (F) when it satisfies

(F) if $x \in U$ and U is open, then there exists an open $V = V(x, U)$ containing x such that $x \in W \subseteq U$ for some $W \in W(y)$ whenever $y \in V$.

Any topological space clearly has such a family of open sets satisfying (F). All the spaces we discuss are determined by placing restrictions on $\mathscr{W}$. One important subclass of conditions requires each $W(x)$ to be countable and deserves a name. Let N denote the set of positive integers, and suppose that $W(N, x) = \{W(n, x): n \in N\}$ is for each $x \in X$ a family of subsets of X containing x and that $\mathscr{W}(N) = \{W(N, x): x \in X\}$. We say that $\mathscr{W}(N)$ satisfies (G) when it satisfies

(G) if $x \in U$ and U is open, then there exists an open $V = V(x, U)$ containing x and such that $x \in W(s, y) \subseteq U$ for some $s \in N$ whenever $y \in V$.

We shall also say that $\mathscr{W}$ satisfies open (F), or neighbourhood (F), or chain (F), if $\mathscr{W}$ satisfies (F) and each element of each $W(x)$ is open, or is a neighbourhood of x, or each $W(x)$ is a chain with respect to inclusion. We similarly modify (G) and, in particular, say that $\mathscr{W}(N)$ satisfies decreasing (G) if $\mathscr{W}(N)$ satisfies (G) and $W(n+1, x) \subseteq W(n, x)$ for each $n \in N$ and each $x \in X$.

Received by the editors January 30, 1984 and, in revised form, June 16, 1984.
1980 *Mathematics Subject Classification*. Primary 54D18, 54E25, 54E35; Secondary 54D15, 54E30, 54E65,
[1]G. M. Reed wishes to acknowledge support as a Futuretrends Fellow of the Institute of Medicine and Mathematics.

Some of our results have already been announced in [3]; in particular, the following, which is the main result of the present paper:

THEOREM. *If X has $\mathscr{W}(N)$ satisfying* open *decreasing* (G), *then X is metrisable.*

X is not, in general, metrisable (Example 3.4) if $\mathscr{W}(N)$ satisfies only *neighbourhood* decreasing (G). Since the integer s in (G) depends on $y \in V$ as well as on x and U, this provokes an interesting comparison with Theorem 1 of [3]. That theorem states that $\mathscr{W}(N)$ satisfying *neighbourhood* decreasing (G) implies metrisability provided s depends only on x and U (and does not vary with y). We also derive two criteria for paracompactness (Theorems 4 and 5) by strengthening chain (F), and we show (Theorem 6) that a first countable space with a family $\mathscr{W}(N)$ satisfying decreasing (G) is semimetric.

We emphasize now that the conditions (F) and (G) with their various modifications are all hereditary and that all the properties we deduce from them alone are hereditary for the spaces considered. Also, all our spaces will satisfy the T_1 separation axiom. A° will denote the interior of the set A.

The paper is arranged roughly in the order of severity of conditions (with weakest first). Each section has two parts: the first consisting in results, the second in examples.

2. When $\mathscr{W}$ does not necessarily consist of chains. As we have already remarked, every topological space X has $\mathscr{W}$ satisfying open (F): one just puts, for each x in X, $W(x) = \{U \subseteq X\colon U \text{ is open and } x \in U\}$. We can, however, impose restrictions on X by bounding the cardinality of the elements of $\mathscr{W}$. When we do this there are some interesting results which can be proved without further restrictions.

THEOREM 1. *If X is separable and has $\mathscr{W}(N)$ satisfying* (G), *then X is hereditarily separable.*

PROOF. Suppose that D is a countable dense subset and Y is any subspace of a space X. Form a set D' by picking one element from each nonempty member of $\{W(n, x) \cap Y\colon x \in D, n \in N\}$. One can easily deduce that D' is dense in Y.

The next result was proved in [3].

THEOREM 2. *If X is separable and has $\mathscr{W}(N)$ satisfying open* (G), *then X is second countable.*

Both Theorems 1 and 2 can easily be extended to the cases of higher cardinalities.

It is easy to see that if $\mathscr{W}(N)$ satisfies neighbourhood (G), then each $W(N, x)$ is a neighbourhood basis at x. The next result establishes a partial converse.

LEMMA 1. *If the first countable space X has $\mathscr{W}(N)$ satisfying* (G), *then there exists $\mathscr{W}'(N)$ satisfying neighbourhood* (G) *in X.*

PROOF. If, for each x in X, $\{V(n, x)\colon n \in N\}$ is a local basis at x, then $\mathscr{W}'(N)$, with elements $W'(x) = \{W(m, x) \cup V(n, x)\colon m, n \in N\}$ (any enumeration), satisfies neighbourhood (G).

EXAMPLES 1. The "bow-tie" space [5], see Example 3.4 below, is not a second countable space but has $\mathscr{W}(N)$ satisfying neighbourhood (G) and is separable. Thus Theorem 2 cannot be extended to the case of neighbourhood (G), and Lemma 1 cannot be extended to establish open (G).

One should note that the topology of a space having $\mathscr{W}$ satisfying (F) is *not* determined by $\mathscr{W}$ (see Example 3.6 below).

3. When $\mathscr{W}$ satisfies chain (F). In [3] it was established that a space having $\mathscr{W}$ satisfying chain neighbourhood (F) is necessarily monotonically normal (and hence collectionwise normal). Essentially the same proof carries over to the case where each $W(x)$ is an arbitrary chain of sets.

THEOREM 3. *If the space X has $\mathscr{W}$ satisfying chain* (F), *then X is monotonically normal.*

PROOF. We must prove that, for each x in an open set U, there is an open set $V(x, U)$ containing x and such that

(a) $x \in U \subseteq U' \Rightarrow V(x, U) \subseteq V(x, U')$,

(b) $x \neq y \Rightarrow V(x, X - \{y\}) \cap V(y, X - \{x\}) = \varnothing$.

These conditions are satisfied by setting $V(x, U) = \{y \in X\colon \exists W \in W(y)$ with $x \in W \subseteq U\}^\circ$.

THEOREM 4. *If the space X has $\mathscr{W}$ satisfying chain* (F) *and each chain $W(x)$ is well ordered by $\supseteq$, then X is paracompact.*

PROOF. By Theorem 3 and the theorem of E. Michael [6] and K. Nagami [8], it will be sufficient to show that X is metacompact. So, suppose $\mathscr{U}$ is an open covering of X, indexed by some ordinal α. For any point x of X and open U containing x, write $V(x, U)$ for an open set found by using (F); so that, there is $W \in W(y)$ with $x \in W \subseteq U$ whenever $y \in V(x, U)$. For each $\beta < \alpha$, define

$$V_\beta = \bigcup\{V(x, U_\beta)\colon x \in U_\beta - \bigcup\{U_\gamma\colon \gamma < \beta\}\}.$$

Clearly $\{V_\beta\colon \beta < \alpha\}$ is a family of open sets such that $V_\beta \subseteq U_\beta$ and $\bigcup\{V_\gamma\colon \gamma < \beta\} = \bigcup\{U_\gamma\colon \gamma < \beta\}$ for each $\beta < \alpha$. Hence $\mathscr{V} = \{V_\beta\colon \beta < \alpha\}$ is a refinement of $\mathscr{U}$ and it only remains to show that $\mathscr{V}$ is point-finite. If $\mathscr{V}$ did not have this property there would exist an x and an increasing sequence $\{\alpha_i\colon i = 1, 2, 3, \dots\}$ of ordinals such that $x \in V_{\alpha_i}$ for each i. So, there are points y_i in $U_{\alpha_i} - \bigcup\{U_\beta\colon \beta < \alpha_i\}$ such that $x \in V(y_i, U_{\alpha_i})$. Hence, for each i, there exists $W_i \in W(x)$ with $y_i \in W_i \subseteq U_{\alpha_i}$. But $y_{i+1} \notin W_i$; for $y_{i+1} \notin U_{\alpha_i}$. Hence W_i is contained in, but is not equal to, W_{i+1}. $\{W_i\colon i = 1, 2, 3, \dots\}$ is an infinite descending chain in $W(x)$, which is well ordered by $\supseteq$. This contradiction establishes Theorem 4.

THEOREM 5. *If the space X has $\mathscr{W}$ satisfying chain neighbourhood* (F), *then X is paracompact.*

PROOF. Once again it will be sufficient to show that X is metacompact. So, suppose that λ is an ordinal and that $\{U_\alpha\colon \alpha < \lambda\} = \mathscr{U}$ is an open covering of X. For each $\alpha < \lambda$ let $X_\alpha = U_\alpha - \bigcup_{\beta<\alpha} U_\beta$, and for each $x \in X_\alpha$ choose $W_x \in W(x)$ with

$W_x \subseteq U_\alpha$. Index X by some ordinal κ : $X = \{x_\beta\colon \beta < \kappa\}$. For each $\beta < \kappa$ we construct a subset Y_β of X: suppose that Y_γ has been chosen for each $\gamma < \beta$; then

$$Y_\beta = \begin{cases} \varnothing & \text{if } x_\beta \in \bigcup_{\gamma<\beta} Y_\gamma, \\ \{y \in X - \bigcup_{\gamma<\beta} Y_\gamma\colon x_\beta \in W_y\} & \text{otherwise.} \end{cases}$$

For each $x \in X$ there is some $\beta < \kappa$ with $x \in Y_\beta$. If $x = x_\beta$, let $V_x = W_x^\circ$; if $x \neq x_\beta$, let $V_x = W_x^\circ - \{x_\beta\}$. Then define $Z_x = V(x, V(x, V_x))$.

For each $\alpha < \lambda$, define $Z_\alpha = \bigcup\{Z_x\colon x \in X_\alpha\}$. Certainly Z_α is an open subset of U_α and $X = \bigcup\{Z_\alpha\colon \alpha < \lambda\}$. Thus it will be sufficient to show that $\{Z_\alpha\colon \alpha < \lambda\}$ is point finite.

Suppose that $x \in Z_\alpha$ for infinitely many α, say $\alpha_1 < \alpha_2 < \alpha_3 < \cdots$. There is $y_n \in X_{\alpha_n}$ with $x \in Z_{y_n}$; and $y_n \in Y_{\beta_n}$ for some $\beta_n < \kappa$. We can assume that either (a) or (b) below holds:

(a) there is $\beta < \kappa$ with $\beta_n = \beta$ for all n;

(b) $\beta_1 < \beta_2 < \beta_3 < \cdots$.

Suppose case (a). Choose $n < m$ so that $y_n \neq x_\beta$ and $y_m \neq x_\beta$. Since $x \in V(y_n, V(y_n, V_{y_n}))$ and $x \in V(y_m, V(y_m, V_{y_m}))$ we either have $y_n \in V(y_m, V_{y_m})$ or $y_m \in V(y_n, V_{y_n})$. But $y_m \notin U_{\alpha_n}$ and $V(y_n, V_{y_n}) \subseteq U_{\alpha_n}$, so $y_n \in V(y_m, V_{y_m})$. There is thus some $W \in W(y_n)$ with $y_m \in W \subseteq V_{y_m}$. Since $W(y_n)$ is a chain and $y_m \notin W_{y_n}$ (as $y_m \notin U_{\alpha_n}$), we know that $W_{y_n} \subseteq W$. But $x_\beta \notin W$ as $x_\beta \notin V_{y_m}$, and $x_\beta \in W_{y_n}$ by assumption. This gives a contradiction.

Suppose case (b). Again suppose $n < m$. As in case (a), there is some $W \in W(y_n)$ with $y_m \in W \subseteq V_{y_m}$ and, by definition of the Y_β's, $x_{\beta_n} \notin W_{y_m}$. But $V_{y_m} \subseteq W_{y_m}$, so $x_{\beta_n} \notin W$. Since $W(y_n)$ is a chain and $y_m \notin W_{y_n}$, we have $W_{y_n} \subseteq W$. However, $x_{\beta_n} \in W_{y_n}$ by definition. This contradicts case (b), and so completes the proof.

EXAMPLES 2. It is easy to see that the ordering of each $W(x)$ is necessary for the results of this section to be true. For example, every second countable space (indeed, every space with a point-countable basis) has $\mathscr{W}(N)$ satisfying open (G); but there are many examples of such spaces which are not collectionwise normal.[2]

The open ordinal space $[0, \omega_1)$ provides an example of a (monotonically normal) space having $\mathscr{W}$ satisfying chain (F) but which is not metacompact (one can put $W(x) = \{[x, \alpha]\colon x \leqslant \alpha < \omega_1\}$). Thus we cannot simultaneously drop the well-ordering assumption of Theorem 4 and the neighbourhood assumption of Theorem 5, if we wish to prove paracompactness.[3]

4. When $\mathscr{W}(N)$ satisfies decreasing (G). We note first that the condition given in the title of this section is more restrictive than stipulating that each $W(N, x)$ be a chain. The spaces considered here are, by the results of §3, (hereditarily) paracompact and monotonically normal. The following lemma is important for the proof of Theorem 6 (which is a more powerful result).

---

[2] *Added in proof.* It is an interesting problem as to whether every space having $\mathscr{W}(N)$ satisfying open (G) also has a point-countable basis.

[3] *Added in proof.* We have derived a more general result from which Theorems 4 and 5 can be deduced (to appear).

LEMMA 2. *Suppose that X is a first countable space having $\mathscr{W}(N)$ satisfying decreasing* (G). *Then X has a G_δ-diagonal.*

PROOF. It is sufficient to construct a sequence $\{\mathscr{G}(n)\colon n = 1, 2, 3, \dots\}$ of open coverings of X such that, for any two distinct points a and b of X, there exists an integer n for which the point pair $\{a, b\}$ is not contained in any element of $\mathscr{G}(n)$.

Suppose that, for each $x \in X$, the sequence $\{R(n, x)\colon n = 1, 2, 3, \dots\}$ is a decreasing local basis (of open sets) at x. For each $x \in X$ and each $n \in N$, we now define, by induction, open sets $H(n, x) \subseteq X$, open coverings $\mathscr{G}(n)$ of X, and functions $p_n\colon \mathscr{G}(n) \to X$.

For each $x \in X$, put $H(1, x) = R(1, x)$. Suppose that $H(m, x)$ has been defined for $m = 1, 2, \dots, n$. Let $\mathscr{G}(n)$ be a locally finite open refinement of $\{V(x, H(n, x))\colon x \in X\}$ (where $V(x, U)$ is chosen by (G) relative to x and U). For $G \in \mathscr{G}(n)$, choose $p = p_n(G)$ to satisfy $G \subseteq V(p, H(n, p))$. For $x \in X$, we now define

$$H(n+1, x) = R(n+1, x) - \Bigg(\bigcup\Big\{\overline{G} : G \in \bigcup_{m \leqslant n} \mathscr{G}(m), x \notin \overline{G}\Big\} \cup \big\{p_m(G)\colon m \leqslant n, G \in \mathscr{G}(m), x \neq p_m(G), x \in \overline{G}\big\}\Bigg),$$

which is open as each $\mathscr{G}(m)$ is locally finite.

If X does not have a G_δ-diagonal, there are distinct a and b in X such that, for every $n \in N$, there exists $G_n \in \mathscr{G}(n)$ with $\{a, b\} \subseteq G_n$. Put $q_n = p_n(G_n)$ for these G_n.

As the local bases are decreasing, there exists $n \in N$ with $R(n, a) \cap R(n, b) = \varnothing$. We may choose $m \in N$ such that $W(m, a) \subseteq R(n, a)$ and $W(m, b) \subseteq R(n, b)$. For one of a and b, say a, there is thus an infinite subset $I \subseteq N$ with $q_i \notin W(m, a)$ for every $i \in I$. Since $a \in G_i \subseteq V(q_i, H(i, q_i))$, there exists $s_i \in N$ with $q_i \in W(s_i, a) \subseteq H(i, q_i)$. And, since $q_i \notin W(m, a)$ for $i \in I$, $s_i < m$ for $i \in I$. We may therefore suppose that all the s_i are the same for $i \in I$.

As $q_i \notin W(m, a)$, $q_i \neq a$ for any $i \in I$. Since I is infinite, $\bigcap\{R(j, q_i)\colon j \in I\} = \{q_i\}$. Thus, since $a \in H(i, q_i) \subseteq R(i, q_i)$ for all $i \in I$, there must be $i < j$ in I with $q_i \neq q_j$. If $q_j \notin \overline{G}_i$, then, by our definition, $H(j, q_j) \cap \overline{G}_i = \varnothing$ as $i < j$, contradicting $a \in H(j, q_j) \cap G_i$. So, $q_j \in \overline{G}_i$, $i < j$, and, by the definition of the $H(n, x)$, $q_i \notin H(j, q_j)$. We have thus contradicted $q_i \in W(s_i, a) = W(s_j, a) \subseteq H(j, q_j)$ since $s_i = s_j$.

THEOREM 6. *If X is a first countable space having decreasing* (G), *then X is semimetric.*

PROOF. Using Theorem 11 of [3], it is sufficient to prove that each x in X has a local basis $\{T(n, x)\colon n = 1, 2, 3, \dots\}$ which satisfies the condition:

(+) for each $x \in X$ and $n \in N$, there exists an open V containing x for which $x \in T(n, y)$ whenever $y \in V$.

Using Lemma 2, choose a sequence $\{\mathscr{H}(n)\colon n = 1, 2, 3, \ldots\}$ of open coverings of X for which $\mathscr{H}(1) > \mathscr{H}(2) > \cdots$ and such that, if $x \neq y$, there exists $n \in N$ with $x \notin \mathrm{St}(y, \mathscr{H}(n))$. (Here, $\mathscr{F} > \mathscr{G}$ denotes that $\mathscr{G}$ refines $\mathscr{F}$, and $\mathrm{St}(x, \mathscr{F})$ denotes $\bigcup\{F \in \mathscr{F}\colon x \in F\}$.) For $x \in X$, let $\{R(n, x)\colon n \in N\}$ be a decreasing local basis at x.

First define, for each $x \in X$ and $n \in N$, $S(n, x) = W(m, x)$, where $m \in N$ is minimal such that $W(m, x) \subseteq \mathrm{St}(x, \mathscr{H}(n))$. Arbitrarily small $W(m, x)$ are needed; for, if $x \neq y$ and $y \in W(k, x)$, there exists n such that $y \notin \mathrm{St}(x, \mathscr{H}(n))$ and hence such that $S(n, x) \subseteq W(k, x)$.

Observe that, for each $s \in N$, there exists $n \in N$ such that $S(n, x) \subseteq W(s, x)$. Define, for each $x \in X$ and $n \in N$,

$$\begin{aligned} T(n, x) &= X \quad \text{if } R(m, x) \not\supseteq S(n, x) \text{ for each } m \in N, \\ &= R(m, x) \quad \text{where } m \text{ is maximal} \leqslant n \text{ with respect to } R(m, x) \supseteq S(n, x). \end{aligned}$$

(This exhausts all possibilities.) For each $m \in N$, there exists $s \in N$ such that $W(s, x) \subseteq R(m, x)$ and, using the above observation, there is $n \geqslant m$ with $S(n, x) \subseteq W(s, x)$. Hence $T(n, x) \subseteq R(m, x)$. Thus $\{T(n, x)\colon n = 1, 2, 3, \ldots\}$ is a local basis at x.

Suppose that $x \in X$ and $n \in N$, and find $U \in \mathscr{H}(n)$ such that $x \in U$. Let V be given by (G) relative to x and U. When $y \in V$ there exists $s \in N$ such that

$$x \in W(s, y) \subseteq U \subseteq \mathrm{St}(y, \mathscr{H}(n)).$$

But then, $W(s, y) \subseteq S(n, y) \subseteq T(n, y)$ by construction. Therefore $x \in T(n, y)$ for each y in the open set V, condition $(+)$ is satisfied, and so X is semimetric.

We have seen (Theorems 3 and 6) that a first countable space, which has $\mathscr{W}(N)$ satisfying decreasing (G), is monotonically normal and semimetric. The following result shows that this implication can be reversed.

THEOREM 7. *Suppose that X is a monotonically normal semimetric space. Then X has $\mathscr{W}(N)$ satisfying decreasing* (G).

PROOF. Suppose that for each $x \in U$ with U open, $V(x, U)$ satisfies (a) and (b) as in Theorem 3; suppose also that d is a semimetric for X. For each $x \in X$ and $n \in N$ let $S(n, x) = \{y \in X\colon d(x, y) < 1/n\}$, and define $V(n, x) = V(x, S(n, x)^\circ)$. $\mathscr{W}(N)$ is defined by setting $W(n, x) = \{y \in X\colon x \in V(n, y)\}$ for each $x \in X$ and $n \in N$. We claim that this $\mathscr{W}(N)$ satisfies (G) ($W(n + 1, x) \subseteq W(n, x)$ trivially).

Suppose that $x \in U$ and that U is open. Find n with $S(n, x) \subseteq U$ and set $V = V(n, x)$. Suppose $y \in V$. By construction we have $x \in W(n, y)$; if $z \in W(n, y)$, then $y \in V(n, x) \cap V(n, z)$, so that $x \in S(n, z)^\circ$ or $z \in S(n, x)^\circ$ (for otherwise $V(n, x) \subseteq V(x, X - \{z\})$ and $V(n, z) \subseteq V(z, X - \{x\})$). In either case we have $d(x, z) < 1/n$, establishing the fact that $x \in W(n, y) \subseteq U$, as desired.

Restricting $\mathscr{W}(N)$ to satisfy open decreasing (G) and using a slight modification of the construction used in the proof of Lemma 2 allows the deduction of the following main result of our paper.

THEOREM 8. *If $\mathscr{W}(N)$ satisfies open decreasing* (G), *then X is metrisable.*

PROOF. For each $n \in N$ we inductively define an open neighbourhood $H(n, x)$ of x, an open covering $\mathscr{G}(n)$ of X and a function p_n: $\mathscr{G}(n) \to X$. Put $H(1, x) = W(1, x)$ for each $x \in X$. Supposing $H(n, x)$ to have been defined for each $x \in X$, define $\mathscr{G}(n)$ to be a locally finite open refinement of $\{V(x, H(n, x)): x \in X\}$. For each $G \in \mathscr{G}(n)$, choose $p = p_n(G)$ with $G \subseteq V(p, H(n, p))$. Finally, for $x \in X$, define

$$H(n+1, x) = \Big[W(n+1, x) \cap \Big(\bigcap \big\{W(t, p_m(G)): G \in \mathscr{G}(m), m \leqslant n, t < n, \\ x \in \overline{G} \cap W(t, p_m(G))\big\}\Big)\Big] \\ - \Big[\bigcup \Big\{\overline{G} : G \in \bigcup_{m \leqslant n} \mathscr{G}(m), x \notin \overline{G}\Big\} \\ \cup \big\{p_m(G): G \in \mathscr{G}(m), m \leqslant n, x \in \overline{G}, x \neq p_m(G)\big\}\Big].$$

By Theorem 3, X is monotonically and *a fortiori* collectionwise normal. So, by a theorem of R. H. Bing [**2**], the proof will be complete once we have shown that $\{\mathscr{G}(n): n = 1, 2, 3, \ldots\}$ is a development for X. If this were not the case, there would be $x \in X$, an open U containing x and, for each $n \in N$, an element $G_n \in \mathscr{G}(n)$ with $x \in G_n$ and $G_n - U \neq \varnothing$. Suppose this and let q_n denote $p_n(G_n)$.

As in the proof of Lemma 2, there must be an infinite subset I of N such that $q_i \neq q_j$ for $i \neq j \in I$. By definition of the $H(n, x)$, if $i < j$ in I, $q_j \in \overline{G}_i$ and $q_i \notin H(j, q_j)$.

If $k \in N$, let $I_k = \{i \in I: q_i \in W(k, x)\}$. Since $x \in G_i \subseteq V(q_i, H(i, q_i))$, by (G) there is $s_i \in N$ with $q_i \in W(s_i, x) \subseteq H(i, q_i)$. If $i \in I - I_k$, then $s_i < k$. If $I - I_k$ is infinite, it has an infinite subset J such that all s_i are the same for $i \in J$. If $i < j$ in J, then $q_i \in W(s_i, x) = W(s_j, x) \subseteq H(j, q_j)$. But this contradicts $q_i \notin H(j, q_j)$ for $i < j$ in I. So, $I - I_k$ is finite for all $k \in N$.

Choose $n \in N$ such that $W(n, x) \subseteq U$. There exists $m \in N$ such that $W(m, x) \subseteq V(x, W(n, x))$. Choose $i \in I_m$. Since $q_i \in W(m, x) \subseteq V(x, W(n, x))$, there exists $t \in N$ with $x \in W(t, q_i) \subseteq W(n, x)$ and, since $W(t, q_i)$ is open, there exists $k \in N$ with $W(k, x) \subseteq W(t, q_i)$. Choose $j \in I_k$ such that $i < j$ and $t < j$.

Recall that $q_j \in \overline{G}_i$ $(G_i \in \mathscr{G}(i))$ whenever $i < j$ in I. Since $q_j \in W(k, x) \subseteq W(t, q_i) = W(t, p_i(G_i))$ and $t < j$, by definition, $H(j, q_j) \subseteq W(t, q_i)$ and, since $W(t, q_i) \subseteq W(n, x) \subseteq U$, we have a contradiction to $G_j - U \neq \varnothing$ and the proof of Theorem 8 is complete.

Theorem 8 may also be proved by showing that X has a point-countable basis and then using R. W. Heath's theorem [**4**] that a semimetric space with a point-countable basis is developable before using Bing's theorem. Z. Balogh [**1**] has recently been able to deduce Theorem 8 from his interesting result that a space having countable pseudo-character and $\mathscr{W}(N)$ satisfying decreasing (G) is stratifiable. (This result strengthens our Theorem 6 by weakening "first countable" to "countable pseudo-character". In the context of first countable monotonically normal spaces "stratifiable" and "semimetric" are equivalent.) A slight sharpening of the proof of Theorem 7 establishes the converse of Balogh's result.

EXAMPLES 3. We first give two examples to show that Lemma 2 cannot be strengthened in certain directions.

EXAMPLE 3.1. Take the ordinal space $[0, \omega_1]$ in which all countable ordinals have been scattered. This (non-first-countable) space can be given $\mathscr{W}(N)$ satisfying decreasing (G):

$$W(1, x) = [x, \omega_1], \qquad W(n, x) = \{x\} \quad \text{if } n > 1.$$

However, this space does not have a G_δ-diagonal.

EXAMPLE 3.2. This example demonstrates that the decreasing assumption in Lemma 2 is required. For each $\alpha \leqslant \omega$, define

$$L_\alpha = \{f\colon (\alpha + 1) \to R\colon f(\beta) \in Q \text{ for all } \beta < \alpha, \text{ but } f(\alpha) \in R - Q\}.$$

(Here R denotes the set of all real numbers and Q denotes the set of all rational numbers.) Our space is $L = \bigcup_{\alpha \leqslant \omega} L_\alpha$ given the order topology with respect to the lexicographical ordering.

This space can be given $\mathscr{W}$ satisfying open (F) with each $W(x)$ a countable, well-ordered chain. (Thus it can be given $\mathscr{W}(N)$ satisfying open (G).) If $f \in L_\alpha$ and $\gamma \leqslant \alpha$, define $f \restriction \gamma$ to be the restriction of f to γ. For each $n \in N$, $\gamma \leqslant \alpha$, $f \in L_\alpha$, define

$$W(n, \gamma, f) = \left\{ g \in \bigcup_{\beta \geqslant \gamma} L_\beta\colon |f(\gamma) - g(\gamma)| < \frac{1}{n} \;\&\; f \restriction \gamma = g \restriction \gamma \right\}.$$

$W(f)$ is then defined to be $\{W(n, \gamma, f)\colon n \in N \;\&\; \gamma \leqslant \alpha\}$. It is not too hard to see, however, that L does not have a G_δ-diagonal (even though it is trivially first countable).

EXAMPLE 3.3. The Michael line [7] provides an easier example of a nonsemimetric space with $\mathscr{W}$ satisfying open (F) with each $W(x)$ a well-ordered, countable chain (see [3]). It also provides an example to show that the possession of $\mathscr{W}$ satisfying chain (F) is not a productive property (as its product with the set of irrationals is not normal).

EXAMPLE 3.4. The "bow-tie" space of L. F. McAuley [5] provides an example of a space with $\mathscr{W}(N)$ satisfying neighbourhood decreasing (G) which is not metrisable. With $X = R \times R$ we set

$$W(n, \langle x, y\rangle) = S(n, \langle x, y\rangle) \quad \text{if } y \neq 0 \text{ and } S(n, \langle x, y\rangle) \text{ does not intersect the } x\text{-axis},$$

$$W(n, \langle x, y\rangle) = S(n, \langle x, y\rangle) \cup \{\langle z, 0\rangle\colon |x - z| < n^{-1/2}\} \quad \text{if } y \neq 0 \text{ but } S(n, \langle x, y\rangle) \text{ does intersect the } x\text{-axis},$$

$$W(n, \langle x, 0\rangle) = \{\langle z, y\rangle\colon y \neq 0 \;\&\; |(x - z)/y| > n \;\&\; |x - z| < 1/n\} \cup \{\langle z, 0\rangle\colon |z - x| < n^{-1/2}\}.$$

(Here $S(n, p)$ is the usual open disc of radius $1/n$ about p.) Note that the $W(n, p)$ are just the usual local bases for X with pieces of the x-axis added.

As was remarked in the Introduction, this example demonstrates that a space with $\mathscr{W}(N)$ satisfying neighbourhood decreasing (G) need not have $\mathscr{W}'(N)$ satisfying neighbourhood decreasing (G′), where (G′) is the stronger variant of (G), in which s is not allowed to vary with $y \in V$ (already discussed as (A) in [3]):

(G′) if $x \in U$ and U is open, then there is some open V containing x and some integer $s = s(x, U)$ such that $x \in W(s, y) \subseteq U$ whenever $y \in V$.

This provides an interesting contrast to the other cases of decreasing (G). We know (Theorems 3, 6 and 7) that any first countable space with $\mathscr{W}(N)$ satisfying decreasing (G) has $\mathscr{W}'(N)$ satisfying decreasing (G′), since the integer n in the proof of Theorem 7 does not depend upon y. Also any space with $\mathscr{W}(N)$ satisfying open decreasing (G) is metrisable (Theorem 8), but every metric space trivially has $\mathscr{W}'(N)$ satisfying open decreasing (G′).

EXAMPLE 3.5. A rather more involved variant of 3.4 can be used to show that Lemma 1 does not carry over to the case of decreasing (G). The following is a first countable space with $\mathscr{W}(N)$ satisfying decreasing (G) but with no $\mathscr{W}(N)$ satisfying neighbourhood decreasing (G). Setting $R^+ = \{x \in R\colon x > 0\}$ and $Q^+ = \{q \in Q\colon q > 0\}$, we take $X = R \times (R^+ \cup \{0\})$. Each element of $\{\langle x, y\rangle\colon y \neq 0\}$ is scattered. When $q \in Q$, a local neighbourhood basis at $\langle q, 0\rangle$ is given by

$$R(n, \langle q, 0\rangle) = \{\langle x, y\rangle\colon y \leqslant |x - q| \leqslant 1/n\}.$$

To construct local neighbourhood bases at the points $\langle x, 0\rangle$ where $x \notin Q$, we choose a bijection θ from the set of irrational numbers to the set of functions $(Q \times N) \to Q^+$, and for each irrational x a sequence $\{q(n, x)\colon n = 1, 2, 3, \ldots\}$ of rationals which converges to x. Given these, we choose for each $x \in R - Q$ a continuous function $g_x\colon R \to R$ with the properties:

(i) $g_x(y) \leqslant |y|$, $g_x(y) = g_x(-y)$, $0 < y \leqslant z \Rightarrow 0 < g_x(y) \leqslant g_x(z)$;

(ii) for each i, $g_x(q(i, x) - x) < \theta(x)(q(i, x), n)$, where n is minimal with respect to $|q(i, x) - x| < 1/n$.

For each $x \in R - Q$, we can now define local neighbourhood bases:

$$R(n, \langle x, 0\rangle) = \{\langle y, z\rangle\colon |x - y| < 1/n \;\&\; z \leqslant g_x(x - y)\}.$$

Our topology can easily be given $\mathscr{W}(N)$ satisfying decreasing (G):

$$W(n, \langle x, y\rangle) = \{\langle x, y\rangle\} \cup \{\langle z, 0\rangle\colon |x - z| < 1/n\} \quad \text{if } y \neq 0 \text{ and } y \leqslant 1/n,$$

$$W(n, \langle x, y\rangle) = \{\langle x, y\rangle\} \quad \text{if } y \neq 0 \text{ and } y > 1/n,$$

$$W(n, \langle x, 0\rangle) = \{\langle z, 0\rangle\colon |x - z| < 1/n\}.$$

If $\mathscr{W}(N)$ were to satisfy *neighbourhood* decreasing (G) for our space, we could construct $f\colon (Q \times N) \to Q^+$ as follows: $f(q, n) = 1/m$, where m is minimal with respect to $R(m, \langle q, 0\rangle) \subseteq W(s, \langle q, 0\rangle)$ for all s such that $R(n, \langle q, 0\rangle) \not\subseteq W(s, \langle q, 0\rangle)$.

Set $p = \langle \theta^{-1}(f), 0\rangle$, $U = R(1, p)$ and $V = V(p, U)$. V contains some point p' of the form $\langle q(i, \theta^{-1}(f)), 0\rangle$; but it is easily shown that there can be no $s \in N$ such that $p \in W(s, p') \subseteq U$, giving the desired contradiction.

EXAMPLE 3.6. It is interesting to note that a family $\mathscr{W}(N)$ satisfying decreasing (G) does not necessarily uniquely determine the topology on a space. For one example of this, note that the $\mathscr{W}(N)$ given in 3.5 works independently of the particular g_x chosen. For a simpler example, consider $X = N \cup \{a\}$ $(a \notin N)$. Now let

$$\begin{aligned} W(1, n) &= \{a, n\} && \text{for all } n \in N, \\ W(m, n) &= \{n\} && \text{for all } n \in N \text{ and } m > 1, \\ W(m, a) &= \{a\} && \text{for all } m \in N. \end{aligned}$$

It is easy to see that this $\mathscr{W}(N)$ satisfies decreasing (G) whether X is given the discrete topology or the topology in which elements of N are scattered and a set containing a is open if and only if it is cofinite. (These topologies are, of course, both metrisable.)

REFERENCES

1. Z. Balogh, *Topological spaces with point-networks*, preprint.

2. R. H. Bing, *Metrization of topological spaces*, Canad. J. Math. **3** (1951), 175–186.

3. P. J. Collins and A. W. Roscoe, *Criteria for metrisability*, Proc. Amer. Math. Soc. **90** (1984), 631–640.

4. R. W. Heath, *On spaces with point-countable bases*, Bull. Acad. Polon. Sci. Sér. Math. Astronom. Phys. **13** (1965), 393–395.

5. L. F. McAuley, *A relation between perfect separability, completeness and normality in semimetric spaces*, Pacific J. Math. **6** (1956), 315–326.

6. E. Michael, *Point-finite and locally finite coverings*, Canad. J. Math. **7** (1955), 275–279.

7. _____, *The product of a normal space and a metric space need not be normal*, Bull. Amer. Math. Soc. **69** (1963), 375–376.

8. K. Nagami, *Paracompactness and screenability*, Nagoya Math. J. **8** (1955), 83–88.

ST. EDMUND HALL, OXFORD, OX1 4AR, UNITED KINGDOM

DEPARTMENT OF MATHEMATICS, OHIO UNIVERSITY, ATHENS, OHIO 45701

UNIVERSITY COLLEGE, OXFORD, OX1 4BH, UNITED KINGDOM

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706