A REGULAR COUNTEREXAMPLE TO THE $\gamma$-SPACE CONJECTURE

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Abstract. This paper presents a completely regular counterexample to the conjecture that every $\gamma$-space is quasi-metrizable. Junnila has shown that developable $\gamma$-spaces are quasi-metrizable; this example shows that "developable" cannot be replaced by "quasi-developable". In the process we provide a method for constructing non-$n$-pretransitive spaces.

1. Introduction. The $\gamma$-space conjecture is the conjecture that every $\gamma$-space is quasi-metrizable. This conjecture has been proven for various classes of spaces, among them spaces with orthobases, developable spaces, and suborderable spaces [G-K2; J2; B-K1]. In [F] it is shown how to construct a counterexample $\hat{X}$ to the $\gamma$-space conjecture from a $\gamma$-space $X$ having a neighbournet $U$ with the property that $U_k$ is not a normal neighbournet for any $k \in \mathbb{N}$ (such a space $X$ may be obtained by taking the topological sum of a sequence of $\gamma$-spaces $X_n$, where $X_n$ is not $n$-pretransitive). However, even if $X$ is completely regular, $\hat{X}$ need not be regular at all. A sufficient condition for $\hat{X}$ to be completely regular is that $X$ be completely regular and that $U_k$ be a clopen neighbournet for every $k \in \mathbb{N}$.

We present here a construction which begins with a quasi-metrizable space $Y$ of cardinality at most $c$ which is not $n^+\text{-pretransitive}$, and yields another quasi-metrizable space $\hat{Y}$ of cardinality $c$ that is not $(n + 1)^+\text{-pretransitive}$. If $U$ is a neighbournet on $Y$ such that $U^{n^+}$ is not normal, then this construction yields a neighbournet $\hat{U}$ on $\hat{Y}$ such that $\hat{U}^{(n+1)^+}$ is not normal (see Lemma 1 below). This construction will allow us to inductively generate a sequence of spaces $X_n = \hat{X}_{n-1}$ with neighbournets $U_n = \hat{U}_{n-1}$ such that $U_n^{n^+}$ is not normal. If we take $X$ to be the topological sum $X = \bigcup_{n=0}^{\infty} X_n$ and $U = \bigcup_{n=0}^{\infty} U_n$, we may then apply [F] to construct a $\gamma$-space $\hat{X}$ which is not quasi-metrizable.

By starting from a suitable space $X_0$, we can inductively guarantee that $X_n$ will be completely regular and that $U_n^{k^+}$ will be clopen for every $k \in \mathbb{N}$ (see Lemma 3 below). In this case, the counterexample $\hat{X}$ will be completely regular, as intended.

2. Terminology. A quasi-metric is a generalized metric $d$ satisfying the triangle inequality $d(x, z) \leq d(x, y) + d(y, z)$ but not necessarily the symmetry axiom $d(x, y) = d(y, x)$ [N; W]. A space $X$ is said to be quasi-metrizable if it has a
compatible quasi-metric \(d\)—i.e. at each point \(x \in X\) the sets \(B_d(x; \varepsilon) = \{ y : d(x, y) < \varepsilon \}\), for \(\varepsilon > 0\), form a neighbourhood base.

We will use Junnila’s neighbournet notation \([J1]\). A *neighbournet* on a space \(X\) is a binary relation \(V\) such that \(V[x]\) is a neighbourhood of \(x\) for every \(x \in X\). A neighbournet \(V\) is called *open*, *closed* or *clopen* if every \(V[x]\) is open, closed or clopen, respectively. A sequence \(\langle V_n : n \in \mathbb{N} \rangle\) of neighbournets is called *basic* if at each point \(x \in X\) the sets \(V_n[x]\), for \(n \in \mathbb{N}\), form a neighbourhood base; and *normal* if \(V_{n+1}^2 \subseteq V_n\) for each \(n\). A neighbournet is said to be *normal* if it is a member of a normal sequence of neighbournets.

With this terminology, a \(T_1\) space is quasi-metrizable if and only if it has a normal basic sequence of neighbournets \([R; J1]\). Similarly, a \(T_1\) space is a \(\gamma\)-space if and only if it has a sequence \(\langle V_n : n \in \mathbb{N} \rangle\) of neighbournets such that the sequence \(\langle V_n^2 : n \in \mathbb{N} \rangle\) is basic \([H; LF; J1]\). Clearly every quasi-metrizable space is a \(\gamma\)-space.

If \(U\) is a binary relation on a space \(X\) we define a new relation \(U^+\) on \(X\) by \(U^+[x] = \cap \{ U[G] : G \text{ is a neighbourhood of } x \}\). If \(U\) is a neighbournet then \(U^n \subseteq (U^+)^n \subseteq U^{n+1}\) for each nonnegative integer \(n\) \([K1]\). We will write \(U^+\) for \((U^+)^+\).

A space \(X\) is called \(n\)-*pretransitive* \((n^+\text{-pretransitive})\) if whenever \(U\) is a neighbournet on \(X\) then \(U^n (U^+)\) is a normal neighbournet \([FL, p. 191, §6.21; cf. also K1]\). The \(n^+\)-pretransitivity property lies strictly between \(n\)-pretransitivity and \((n+1)\)-pretransitivity. Since \(U^0[x] = \{ x \}\), observe that a space is \(0\)-pretransitive \((0^+\text{-pretransitive})\) if and only if it is discrete (the arbitrary intersection of open sets is open).

The importance of \(n\)- and \(n^+\)-pretransitivity is that an \(n\)- or \(n^+\)-pretransitive \(\gamma\)-space is quasi-metrizable \([FL, p. 165, §7.19]\), and that almost all partial solutions to the \(\gamma\)-space conjecture have implicitly used this property: \([G; J2; K1; K2]\) have all shown that the spaces concerned were \(2\)- or \(2^+\)-pretransitive.

3. The construction of \(\hat{Y}\) and \(\hat{U}\). Let \(Y\) be a quasi-metrizable space and \(\langle V_n : n \in \mathbb{N} \rangle\) a normal basic sequence for \(Y\). The structure of \(\hat{Y}\) is as follows.

The points of \(\hat{Y}\) are the points of \(Y \times \mathbb{R}\). We presume that \(\mathbb{R}\) is partitioned into sets \(A\) and \(B\).

For each \(b \in B\) we declare \(Y \times \{ b \}\) to be a clopen subspace of \(\hat{Y}\) canonically homeomorphic to \(Y\). If \(\langle y, b \rangle \in Y \times \{ b \}\) we define \(\hat{V}_n[\langle y, b \rangle] = V_n[y] \times \{ b \}\).

We presume that \(Z\) is a chosen subset of \(Y \times A\); and for each \(\langle x, a \rangle \in Z\) that \(S(x, a)\) is a chosen subset of \(Y \times B\). We define the basic neighbourhoods of \(\langle x, a \rangle \in Z\) to be

\[
\hat{V}_n[\langle x, a \rangle] = \{ \langle x, a \rangle \} \cup \{ V_n[y] \times \{ b \} : \langle y, b \rangle \in S(x, a) \text{ and } |b - a| < 2^{-n} \}.
\]

All points \(\langle x, a \rangle \in X \times A\) which are not in \(Z\) are isolated; for these points we define \(\hat{V}_n[\langle x, a \rangle] = \{ \langle x, a \rangle \}\).

The structure of \(\hat{Y}\) as outlined above does not by itself guarantee that \(\hat{Y}\) will be Hausdorff, even if \(Y\) is. However, \(\hat{Y}\) will be \(T_1\), and so the Hausdorff property will be guaranteed if \(\hat{Y}\) is regular. It is not difficult to show that \(\langle \hat{V}_n : n \in \mathbb{N} \rangle\) is a normal basic sequence for \(\hat{Y}\); and hence \(\hat{Y}\) is quasi-metrizable.
The structure of $\hat{U}$ is as follows. We presume that $A$ is further partitioned into sets $A_p$ ($p \in \mathbb{N}$), and that $Z_p$ denotes the set of all points $\langle x, a \rangle \in Z$ with $a \in A_p$. Define $\hat{U}$ by

$$
\hat{U}[\langle x, b \rangle] = U[x] \times \{b\} \quad \text{if } b \in B;
$$

$$
\hat{U}[\langle x, a \rangle] = \hat{V}_p[\langle x, a \rangle] \quad \text{if } \langle x, a \rangle \in Z_p, p \in \mathbb{N};
$$

$$
\hat{U}[\langle x, a \rangle] = \{(x, a)\} \quad \text{otherwise}.
$$

Observe that if $\langle x, a \rangle \in Z$, then

$$
\hat{U}^\ast[\langle x, a \rangle] = \hat{U}^\ast[\langle x, a \rangle] \cup \bigcap_{k=1}^\infty \bigcup \{U^n \circ V_k[y] \times \{b\} : \langle y, b \rangle \in S(x, a) \text{ and } |b - a| < 2^{-k}\}
$$

Note that the construction of $\hat{Y}$ and $\hat{U}$ depends on the choices made of $A = \bigcup_{p=1}^\infty A_p$, $B$, $Z$, and $S(x, a)$ for each $\langle x, a \rangle \in Z$. We will elaborate later on how these choices are to be made. The lemmas below discuss the properties required of $\hat{Y}$ and $\hat{U}$.

**Lemma 1.** Suppose that $U^{n+1}$ is not a normal neighbournet on $Y$. If $\hat{Y}$ is constructed so that

(I) $B$ is a dense Baire subset of $\mathbb{R}$;

(II) if $E$ is a subset of $Y \times B$, and the canonical projection of $E$ onto $B$ is somewhere dense in $\mathbb{R}$, then for every $p \in \mathbb{N}$ there is a point $\langle x, a \rangle \in Z_p$ such that $\langle x, a \rangle \in \text{cl}(E \cap S(x, a))$; and

(III) for each $\langle x, a \rangle \in Z$ and each $b \in B$ there is at most one point of $S(x, a)$ in $Y \times \{b\}$;

then $\hat{U}^{n+1}$ will not be a normal neighbournet on $\hat{Y}$.

**Proof.** Let $W$ be a normal neighbournet on $\hat{Y}$. To show that $\hat{U}^{n+1}$ is not normal, we will show that $W^2 \not\subseteq \hat{U}^{n+1}$. Because $U^n$ is not normal, we may find for each $b \in B$ a point $y_b \in Y$ such that $W[\langle y_b, b \rangle] \not\subseteq U^n[\langle y_b \rangle] \times \{b\}$. Let $G_b$ be a neighbourhood of $y_b$ in $Y$ such that $W[\langle y_b, b \rangle] \not\subseteq U^n[G_b] \times \{b\}$.

By (I), we may find a fixed $p \in \mathbb{N}$ and a subset $D$ of $B$ which is somewhere dense in $\mathbb{R}$ such that $V_p[y_b] \subseteq G_b$ for all $b \in D$.

By (II), there exists a point $\langle x, a \rangle \in Z_p$ such that $\langle x, a \rangle \in \text{cl}(\{\langle y_b, b \rangle : b \in D\} \cap S(x, a))$. Note that

$$
\hat{U}^{n+1}[\langle x, a \rangle] = \hat{U}^{n+1}[\langle x, a \rangle] = U^n \circ \hat{V}_p[\langle x, a \rangle] = \{(x, a)\} \cup \bigcup \{U^n \circ V_k[y] \times \{b\} : \langle y, b \rangle \in S(x, a) \text{ and } |b - a| < 2^{-p}\}.
$$

Choose some $\langle y_b, b \rangle \in W[\langle x, a \rangle] \cap S(x, a)$ so that $b \in D$. Then $U^n \circ V_p[y_b] \subseteq U^n[G_b]$ and hence we may find some point $\langle z, b \rangle \in W[\langle y_b, b \rangle \setminus U^n \circ V_p[y_b] \times \{b\}$. Thus $\langle z, b \rangle \in W^2[\langle x, a \rangle]$ and, by (III), $\langle z, b \rangle \not\in \hat{U}^{n+1}[\langle x, a \rangle] = \hat{U}^{n+1}[\langle x, a \rangle]$, as required.
Proposition 2. Suppose \( \langle x, a \rangle \in Z \) has a neighbourhood \( \hat{G} = \{ \langle x, a \rangle \} \cup \bigcup_{i=1}^{\infty} G_i \times \{ b_i \} \), where the \( G_i \) are clopen in \( Y \) and the \( b_i \) converge in \( R \) to \( a \). If \( \langle x, a \rangle \) is the only point of \( Z \) in \( Y \times \{ a \} \), then \( \hat{G} \) is clopen in \( \hat{Y} \).

Lemma 3. Suppose (a) each \( V_n \) is a clopen neighbournet on \( Y \); (b) \( U^k \) is a clopen neighbournet for each \( k \in \mathbb{N} \); and (c) \( U^k \circ V_n \) is a clopen neighbournet for each \( k, n \in \mathbb{N} \). If \( \hat{Y} \) is constructed so that

- (IV) for each \( a \in A \) there is at most one point of \( Z \) in \( Y \times \{ a \} \); and
- (V) for each \( \langle x, a \rangle \in Z \), \( S(x, a) \) is a sequence \( \{ \langle y_i, b_i \rangle : i \in \mathbb{N} \} \) where the \( b_i \) converge in \( R \) to \( a \);

then (a) each \( \hat{V}_n \) is a clopen neighbournet on \( \hat{Y} \); (b) \( \hat{U}^k \) is a clopen neighbournet for each \( k \in \mathbb{N} \); and (c) \( \hat{U}^k \circ \hat{V}_n \) is a clopen neighbournet for each \( k, n \in \mathbb{N} \).

Proof. It will suffice to show for each \( \langle x, a \rangle \in Z \) that \( \hat{V}_n \setminus \{ x, a \} \), \( \hat{U}^k \setminus \{ x, a \} \), and \( \hat{U}^k \circ \hat{V}_n \setminus \{ x, a \} \) are clopen. Suppose \( \langle x, a \rangle \in Z \). By (V), let \( S(x, a) = \{ \langle y_i, b_i \rangle : i \in \mathbb{N} \} \). Then

\[
\hat{V}_n \setminus \{ x, a \} = \{ \langle x, a \rangle \} \cup \{ V_n \setminus \{ y_i \} \times \{ b_i \} : |b_i - a| < 2^{-n} \},
\]

\[
\hat{U}^k \setminus \{ x, a \} = \{ \langle x, a \rangle \} \cup \{ U^{k-1} \circ V_p \setminus \{ y_i \} \times \{ b_i \} : |b_i - a| < 2^{-p} \},
\]

\[
\hat{U}^k \circ \hat{V}_n \setminus \{ x, a \} = \{ \langle x, a \rangle \} \cup \{ U^k \circ V_n \setminus \{ y_i \} \times \{ b_i \} : |b_i - a| < 2^{-n} \}
\]

\[
\cup \{ U^{k-1} \circ V_p \setminus \{ y_i \} \times \{ b_i \} : |b_j - a| < 2^{-p} \}.
\]

The required result now follows from Proposition 2, using (IV) and the assumptions (a) and (c). (Note that if \( k - 1 = 0 \) then \( U^{k-1} \circ V_p \circ V_p \) and so we would use (a) instead of (c) to guarantee that \( U^{k-1} \circ V_p \setminus \{ y_i \} \) was clopen.)

Now let us show that conditions (I) through (V) from Lemmas 1 and 3 may be met by suitably constructing \( \hat{Y} \) from a space \( Y \) of cardinality at most \( c \).

First, we may partition \( R \) into sets \( A \) and \( B \), where \( A \) has cardinality \( c \) on every open interval of \( R \) and \( B \) is dense and Baire in \( R \).

If the cardinality of \( Y \) is no more than \( c \), there will also be no more than \( c \) countable subsets \( E \) of \( Y \times B \). Then by a straightforward transfinite induction, choose for each countable \( E \subseteq Y \times B \) whose canonical projection onto \( B \) is dense in some interval \( (c_1, c_2) \) in \( R \) and for each \( p \in \mathbb{N} \), a distinct real number \( a_{E_p} \in A \cap (c_1, c_2) \). Choose an arbitrary \( x_{E_p} \in Y \) and let \( S(x_{E_p}, a_{E_p}) \) be any sequence of points \( \{ y_i, b_i \} \in E \) where the \( b_i \) are distinct and converge in \( R \) to \( a_{E_p} \).

Let \( Z \) consist of all points \( \langle x_{E_p}, a_{E_p} \rangle \); and partition \( A \) into sets \( A_p (p \in \mathbb{N}) \) so that \( a_{E_p} \in A_p \). Then conditions (I) through (V) are met (note that it is sufficient to prove (II) for countable sets \( E \)).

4. The counterexample. To complete the construction of the counterexample \( \hat{X} \), all that remains to be done is to provide a suitable space \( X_0 \) to start off the induction. For this purpose we will choose the convergent sequence space \( \{ 2^{-k} : k \in \mathbb{N} \} \cup \{ 0 \} \). Observe that \( X_0 \) is not \( 0^+ \)-pretransitive, and in fact \( U_0^{0^+} \) will not be a normal neighbournet no matter what \( U_0 \) is. To satisfy the inductive assumptions (a), (b) and (c) of Lemma 3 we may define the neighbounet \( U_0 \) and the normal basic sequence \( \{ V_n : n \in \mathbb{N} \} \) so that \( U_0[x] = X_0 \) and \( V_n[2^{-k}] = \{ 2^{-k} \} \); \( V_n[0] = \{ 2^{-k} : k > n \} \cup \{ 0 \} \).
The counterexample $\bar{X}$ thereby produced will have the following properties. Note that the construction of $\bar{Y}$ from $Y$ and the construction of $\bar{X}$ from $X$ both preserve scatteredness, and that both increase the Cantor-Bendixson rank of the space by 1. Therefore $\bar{X}$ will be scattered, and the Cantor-Bendixson rank of $\bar{X}$ will be $\omega + 1$. Consequently, $\bar{X}$ will be transitive (by transfinite induction and [FL, 6.16 and 6.17]), hereditarily weakly $\theta$-refinable, and quasi-developable. Junnila has shown in [J2] that developable $\gamma$-spaces are quasi-metrizable; this demonstrates that developable cannot be weakened to quasi-developable.

We remark in passing that, with a modified construction of $\bar{Y}$, a counterexample $\bar{X}$ can be constructed which has the above properties and which is in addition submetrizable—that is, it has normal $G_\delta$-diagonal sequence.

Finally, we observe that the construction given in [F] and used here cannot produce a normal counterexample to the $\gamma$-space conjecture. In particular, if $X$ is any $T_1$ space containing at least 2 points then $\bar{X}$ will not be normal. For example, if $X$ is the 2-point discrete space $\{0, 1\}$ then $\bar{X}$ will consist of levels 1 through $\omega$ inclusive of a Cantor tree with the tree topology, a nonnormal space. To see this for a larger space $X$, consider a 2-point subset of $X$ and the Cantor tree it generates in $\bar{X}$. This raises the question: Are normal $\gamma$-spaces quasi-metrizable?

ADDED IN PROOF. The answer to the last question is no; there exists a paracompact counterexample.

References


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