A REGULAR COUNTEREXAMPLE TO THE $\gamma$-SPACE CONJECTURE

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Abstract. This paper presents a completely regular counterexample to the conjecture that every $\gamma$-space is quasi-metrizable. Junnila has shown that developable $\gamma$-spaces are quasi-metrizable; this example shows that "developable" cannot be replaced by "quasi-developable". In the process we provide a method for constructing non-$n$-pretransitive spaces.

1. Introduction. The $\gamma$-space conjecture is the conjecture that every $\gamma$-space is quasi-metrizable. This conjecture has been proven for various classes of spaces, among them spaces with orthobases, developable spaces, and suborderable spaces [G-K2; J2; B-K1]. In [F] it is shown how to construct a counterexample $\hat{X}$ to the $\gamma$-space conjecture from a $\gamma$-space $X$ having a neighbournet $U$ with the property that $U^k$ is not a normal neighbournet for any $k \in \mathbb{N}$ (such a space $X$ may be obtained by taking the topological sum of a sequence of $\gamma$-spaces $X_n$, where $X_n$ is not $n$-pretransitive). However, even if $X$ is completely regular, $\hat{X}$ need not be regular at all. A sufficient condition for $\hat{X}$ to be completely regular is that $X$ be completely regular and that $U^k$ be a clopen neighbournet for every $k \in \mathbb{N}$.

We present here a construction which begins with a quasi-metrizable space $Y$ of cardinality at most $c$ which is not $n^+\text{-pretransitive}$, and yields another quasi-metrizable space $\hat{Y}$ of cardinality $c$ that is not $(n + 1)^+\text{-pretransitive}$. If $U$ is a neighbournet on $Y$ such that $U^{n^+}$ is not normal, then this construction yields a neighbournet $\hat{U}$ on $\hat{Y}$ such that $\hat{U}^{(n+1)^+}$ is not normal (see Lemma 1 below). This construction will allow us to inductively generate a sequence of spaces $X_n = \hat{X}_{n-1}$ with neighbournets $U_n = \hat{U}_{n-1}$ such that $U_n^{n^+}$ is not normal. If we take $X$ to be the topological sum $X = \bigcup_{n=0}^{\infty} X_n$ and $U = \bigcup_{n=0}^{\infty} U_n$, we may then apply [F] to construct a $\gamma$-space $\hat{X}$ which is not quasi-metrizable.

By starting from a suitable space $X_0$, we can inductively guarantee that $X_n$ will be completely regular and that $U_n^{n^+}$ will be clopen for every $k \in \mathbb{N}$ (see Lemma 3 below). In this case, the counterexample $\hat{X}$ will be completely regular, as intended.

2. Terminology. A quasi-metric is a generalized metric $d$ satisfying the triangle inequality $d(x, z) \leq d(x, y) + d(y, z)$ but not necessarily the symmetry axiom $d(x, y) = d(y, x)$ [N; W]. A space $X$ is said to be quasi-metrizable if it has a

Received by the editors May 9, 1983 and, in revised form, July 16, 1984.
1980 Mathematics Subject Classification. Primary 54E15; Secondary 54G20, 54D15, 54E30.
Key words and phrases. Quasi-metric, $\gamma$-space, $n$-pretransitive space, normal neighbournet, quasi-developable, scattered space.

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0002-9939/85 $1.00 + .25 per page
compatible quasi-metric $d$—i.e. at each point $x \in X$ the sets $B_d(x; \epsilon) = \{ y: d(x, y) < \epsilon \}$, for $\epsilon > 0$, form a neighbourhood base.

We will use Junnila's neighbournet notation [J1]. A neighbournet on a space $X$ is a binary relation $V$ such that $V[x]$ is a neighbourhood of $x$ for every $x \in X$. A neighbournet $V$ is called open, closed or clopen if every $V[x]$ is open, closed or clopen, respectively. A sequence $\langle V_n: n \in \mathbb{N} \rangle$ of neighbournets is called basic if at each point $x \in X$ the sets $V_n[x]$, for $n \in \mathbb{N}$, form a neighbourhood base; and normal if $V_{n+1}^2 \subseteq V_n^2$ for each $n$. A neighbournet is said to be normal if it is a member of a normal sequence of neighbournets.

With this terminology, a $T_1$ space is quasi-metrizable if and only if it has a normal basic sequence of neighbournets [R; J1]. Similarly, a $T_1$ space is a $\gamma$-space if and only it has a sequence $\langle V_n: n \in \mathbb{N} \rangle$ of neighbournets such that the sequence $\langle V_n^2: n \in \mathbb{N} \rangle$ is basic [H; LF; J1]. Clearly every quasi-metrizable space is a $\gamma$-space.

If $U$ is a binary relation on a space $X$ we define a new relation $U^+$ on $X$ by $U^+[x] = \cap \{ U[G]: G$ is a neighbourhood of $x \}. If U is a neighbournet then $U^n \subseteq (U^n)^+ \subseteq U^{n+1}$ for each nonnegative integer $n$ [K1]. We will write $U^n+$ for $(U^n)^+$.

A space $X$ is called $n$-pretransitive ($n^+$-pretransitive) if whenever $U$ is a neighbournet on $X$ then $U^n (U^n+)$ is a normal neighbournet [FL, p. 191, §6.21; cf. also K1]. The $n^+$-pretransitivity property lies strictly between $n$-pretransitivity and $(n + 1)$-pretransitivity. Since $U^0[x] = \{ x \}$, observe that a space is 0-pretransitive ($0^+$-pretransitive) if and only if it is discrete (the arbitrary intersection of open sets is open).

The importance of $n$- and $n^+$-pretransitivity is that an $n$- or $n^+$-pretransitive $\gamma$-space is quasi-metrizable [FL, p. 165, §7.19], and that almost all partial solutions to the $\gamma$-space conjecture have implicitly used this property: [G; J2; K1; K2] have all shown that the spaces concerned were 2- or 2$^+$-pretransitive.

3. The construction of $\hat{Y}$ and $\hat{U}$. Let $Y$ be a quasi-metrizable space and $\langle V_n: n \in \mathbb{N} \rangle$ a normal basic sequence for $Y$. The structure of $\hat{Y}$ is as follows.

The points of $\hat{Y}$ are the points of $Y \times \mathbb{R}$. We presume that $\mathbb{R}$ is partitioned into sets $A$ and $B$.

For each $b \in B$ we declare $Y \times \{ b \}$ to be a clopen subspace of $\hat{Y}$ canonically homeomorphic to $Y$. If $\langle y, b \rangle \in Y \times \{ b \}$ we define $\hat{V}_n[\langle y, b \rangle] = V_n[y] \times \{ b \}$.

We presume that $Z$ is a chosen subset of $Y \times A$; and for each $\langle x, a \rangle \in Z$ that $S(x, a)$ is a chosen subset of $Y \times B$. We define the basic neighbourhoods of $\langle x, a \rangle \in Z$ to be

$$\hat{V}_n[\langle x, a \rangle] = \{ \langle x, a \rangle \} \cup \bigcup \{ V_n[y] \times \{ b \}: \langle y, b \rangle \in S(x, a) \text{ and } |b - a| < 2^{-n} \}.$$ 

All points $\langle x, a \rangle$ in $X \times A$ which are not in $Z$ are isolated; for these points we define $\hat{V}_n[\langle x, a \rangle] = \{ \langle x, a \rangle \}$.

The structure of $\hat{Y}$ as outlined above does not by itself guarantee that $\hat{Y}$ will be Hausdorff, even if $Y$ is. However, $\hat{Y}$ will be $T_1$, and so the Hausdorff property will be guaranteed if $\hat{Y}$ is regular. It is not difficult to show that $\langle \hat{V}_n: n \in \mathbb{N} \rangle$ is a normal basic sequence for $\hat{Y}$; and hence $\hat{Y}$ is quasi-metrizable.
The structure of $\mathcal{U}$ is as follows. We presume that $A$ is further partitioned into sets $A_p$ ($p \in \mathbb{N}$), and that $Z_p$ denotes the set of all points $\langle x, a \rangle \in Z$ with $a \in A_p$. Define $\mathcal{U}$ by

$$
\mathcal{U}\[(x,b)\] = U[x] \times \{b\} \quad \text{if } b \in B; \\
\mathcal{U}\[(x,a)\] = \hat{V}_p\[(x,a)\] \quad \text{if } \langle x, a \rangle \in Z_p, p \in \mathbb{N}; \\
\mathcal{U}\[(x,a)\] = \{(x,a)\} \quad \text{otherwise}.
$$

Observe that if $\langle x, a \rangle \in Z$, then

$$
\mathcal{U}\n+\[(x,a)\] = \mathcal{U}\n\[(x,a)\] \cup \bigcup_{k=1}^{\infty} \bigcup_{b \in D} \mathcal{U}\n\circ \mathcal{V}_k\[(y)\] \times \{b\} : \langle y, b \rangle \in S(x, a)
$$

and $|b - a| < 2^{-k}$}

Note that the construction of $\mathcal{Y}$ and $\mathcal{U}$ depends on the choices made of $A = \bigcup_{p=1}^{\infty} A_p$, $B$, $Z$, and $S(x, a)$ for each $\langle x, a \rangle \in Z$. We will elaborate later on how these choices are to be made. The lemmas below discuss the properties required of $\mathcal{Y}$ and $\mathcal{U}$.

**Lemma 1.** Suppose that $U^{n+1}$ is not a normal neighbournet on $Y$. If $\mathcal{Y}$ is constructed so that

(I) $B$ is a dense Baire subset of $\mathbb{R}$;

(II) if $E$ is a subset of $Y \times B$, and the canonical projection of $E$ onto $B$ is somewhere dense in $\mathbb{R}$, then for every $p \in \mathbb{N}$ there is a point $\langle x, a \rangle \in Z_p$ such that $\langle x, a \rangle \in \text{cl}(E \cap S(x, a))$; and

(III) for each $\langle x, a \rangle \in Z$ and each $b \in B$ there is at most one point of $S(x, a)$ in $\mathcal{Y} \times \{b\}$;

then $\mathcal{U}^{n+1}$ will not be a normal neighbournet on $\mathcal{Y}$.

**Proof.** Let $W$ be a normal neighbournet on $\mathcal{Y}$. To show that $\mathcal{U}^{(n+1)+}$ is not normal, we will show that $W^2 \not\subseteq \mathcal{U}^{(n+1)+}$.

Because $U^{n+1}$ is not normal, we may find for each $b \in B$ a point $y_b \in Y$ such that $W[\langle y_b, b \rangle] \not\subseteq U^n[\langle y_b \rangle] \times \{b\}$. Let $G_b$ be a neighbourhood of $y_b$ in $Y$ such that $W[\langle y_b, b \rangle] \not\subseteq U^n[G_b] \times \{b\}$.

By (I), we may find a fixed $p \in \mathbb{N}$ and a subset $D$ of $B$ which is somewhere dense in $\mathbb{R}$ such that $V_p[y_b] \subseteq G_b$ for all $b \in D$.

By (II), there exists a point $\langle x, a \rangle \in Z_p$ such that $\langle x, a \rangle \in \text{cl}((\langle y_b, b \rangle : b \in D) \cap S(x, a))$. Note that

$$
\hat{U}^{(n+1)+}[\langle x, a \rangle] = \hat{U}^{n+1}[\langle x, a \rangle] = \hat{U}^n \circ \hat{V}_p[\langle x, a \rangle]
$$

\[= \{(x,a)\} \cup \left\{U^n \circ V_p[\langle y \rangle] \times \{b\} : \langle y, b \rangle \in S(x, a) \text{ and } |b - a| < 2^{-p}\right\}.
$$

Choose some $\langle y_b, b \rangle \in W[\langle x, a \rangle] \cap S(x, a)$ so that $b \in D$. Then $U^n \circ V_p[y_b] \subset U^n[G_b]$ and hence we may find some point $\langle z, b \rangle \in W[\langle y_b, b \rangle] \setminus U^n \circ V_p[y_b] \times \{b\}$. Thus $\langle z, b \rangle \in W^2[\langle x, a \rangle]$ and, by (III), $\langle z, b \rangle \notin \hat{U}^{n+1}[\langle x, a \rangle] = \hat{U}^{(n+1)+}[\langle x, a \rangle]$ as required.
**Proposition 2.** Suppose \( (x, a) \in Z \) has a neighbourhood \( \hat{G} = \{(x, a)\} \cup \bigcup_{i=1}^{\infty} G_i \times \{b_i\} \), where the \( G_i \) are clopen in \( Y \) and the \( b_i \) converge in \( \mathbb{R} \) to \( a \). If \( (x, a) \) is the only point of \( Z \) in \( Y \times \{a\} \), then \( \hat{G} \) is clopen in \( \hat{Y} \).

**Lemma 3.** Suppose (a) each \( V_n \) is a clopen neighbourhood on \( Y \); (b) \( U_k \) is a clopen neighbourhood for each \( k \in \mathbb{N} \); and (c) \( U_k \circ V_n \) is a clopen neighbourhood for each \( k, n \in \mathbb{N} \). If \( \hat{Y} \) is constructed so that

(IV) for each \( a \in A \) there is at most one point of \( Z \) in \( Y \times \{a\} \); and

(V) for each \( (x, a) \in Z \), \( S(x, a) \) is a sequence \( \{(y_i, b_i) : i \in \mathbb{N}\} \) where the \( b_i \) converge in \( \mathbb{R} \) to \( a \);

then (a) each \( \hat{V}_n \) is a clopen neighbourhood on \( \hat{Y} \); (b) \( \hat{U}_k \) is a clopen neighbourhood for each \( k \in \mathbb{N} \); and (c) \( \hat{U}_k \circ \hat{V}_n \) is a clopen neighbourhood for each \( k, n \in \mathbb{N} \).

**Proof.** It will suffice to show for each \( (x, a) \in Z \) that \( \hat{V}_n \backslash \{(x, a)\} \), \( \hat{U}_k \backslash \{(x, a)\} \) and \( \hat{U}_k \circ \hat{V}_n \backslash \{(x, a)\} \) are clopen. Suppose \( (x, a) \in Z \). By (V), let \( S(x, a) = \{(y_i, b_i) : i \in \mathbb{N}\} \). Then

\[
\hat{V}_n \backslash \{(x, a)\} = \{(x, a)\} \cup \{V_n[y_i] \times \{b_i\} : |b_i - a| < 2^{-n}\},
\]

\[
\hat{U}_k \backslash \{(x, a)\} = \{(x, a)\} \cup \{U_k^{-1} \circ V_p[y_i] \times \{b_i\} : |b_i - a| < 2^{-p}\},
\]

\[
\hat{U}_k \circ \hat{V}_n \backslash \{(x, a)\} = \{(x, a)\} \cup \{U_k \circ V_n[y_i] \times \{b_i\} : |b_i - a| < 2^{-n}\}
\]

\[
\bigcup \{U_k^{-1} \circ V_p[y_i] \times \{b_j\} : |b_j - a| < 2^{-p}\}.
\]

The required result now follows from Proposition 2, using (IV) and the assumptions (a) and (c). (Note that if \( k - 1 = 0 \) then \( U_k^{-1} \circ V_p = V_p \), and so we would use (a) instead of (c) to guarantee that \( U_k^{-1} \circ V_p[y_i] \) was clopen.)

Now let us show that conditions (I) through (V) from Lemmas 1 and 3 may be met by suitably constructing \( \hat{Y} \) from a space \( Y \) of cardinality at most \( c \).

First, we may partition \( \mathbb{R} \) into sets \( A \) and \( B \), where \( A \) has cardinality \( c \) on every open interval of \( \mathbb{R} \) and \( B \) is dense and Baire in \( \mathbb{R} \).

If the cardinality of \( Y \) is no more than \( c \), there will also be no more than \( c \) countable subsets \( E \) of \( Y \times B \). Then by a straightforward transfinite induction, choose for each countable \( E \subseteq Y \times B \) whose canonical projection onto \( B \) is dense in some interval \( (c_1, c_2) \) in \( \mathbb{R} \) and for each \( p \in \mathbb{N} \), a distinct real number \( a_{Ep} \in A \cap (c_1, c_2) \). Choose an arbitrary \( x_{Ep} \in Y \) and let \( S(x_{Ep}, a_{Ep}) \) be any sequence of points \( \langle y_i, b_i \rangle \) in \( E \) where the \( b_i \) are distinct and converge in \( \mathbb{R} \) to \( a_{Ep} \).

Let \( Z \) consist of all points \( \langle x_{Ep}, a_{Ep} \rangle \); and partition \( A \) into sets \( A_p \) \( (p \in \mathbb{N}) \) so that \( a_{Ep} \in A_p \). Then conditions (I) through (V) are met (note that it is sufficient to prove (II) for countable sets \( E \)).

**4. The counterexample.** To complete the construction of the counterexample \( \tilde{X} \), all that remains to be done is to provide a suitable space \( X_0 \) to start off the induction. For this purpose we will choose the convergent sequence space \( \{2^{-k} : k \in \mathbb{N}\} \cup \{0\} \).

Observe that \( X_0 \) is not \( 0^+\)-pretransitive, and in fact \( U_0^{0^+} \) will not be a normal neighbourhood no matter what \( U_0 \) is. To satisfy the inductive assumptions (a), (b) and (c) of Lemma 3 we may define the neighbourhood \( U_0 \) and the normal basic sequence \( \langle V_n : n \in \mathbb{N}\rangle \) so that \( U_0[x] = X_0 \) and \( V_n[2^{-k}] = \{2^{-k}\}; \) \( V_n[0] = \{2^{-k} : k > n\} \cup \{0\} \).
The counterexample $\tilde{X}$ thereby produced will have the following properties. Note that the construction of $\tilde{Y}$ from $Y$ and the construction of $\tilde{X}$ from $X$ both preserve scatteredness, and that both increase the Cantor-Bendixson rank of the space by 1. Therefore $\tilde{X}$ will be scattered, and the Cantor-Bendixson rank of $\tilde{X}$ will be $\omega + 1$. Consequently, $\tilde{X}$ will be transitive (by transfinite induction and [FL, 6.16 and 6.17]), hereditarily weakly $\theta$-refinable, and quasi-developable. Junnila has shown in [J2] that developable $\gamma$-spaces are quasi-metrizable; this demonstrates that developable cannot be weakened to quasi-developable.

We remark in passing that, with a modified construction of $\tilde{Y}$, a counterexample $\tilde{X}$ can be constructed which has the above properties and which is in addition submetrizable—that is, it has normal $G_\delta$-diagonal sequence.

Finally, we observe that the construction given in [F] and used here cannot produce a normal counterexample to the $\gamma$-space conjecture. In particular, if $X$ is any $T_1$ space containing at least 2 points then $\tilde{X}$ will not be normal. For example, if $X$ is the 2-point discrete space $\{0, 1\}$ then $\tilde{X}$ will consist of levels 1 through $\omega$ inclusive of a Cantor tree with the tree topology, a nonnormal space. To see this for a larger space $X$, consider a 2-point subset of $X$ and the Cantor tree it generates in $\tilde{X}$. This raises the question: Are normal $\gamma$-spaces quasi-metrizable?

ADDED IN PROOF. The answer to the last question is no; there exists a paracompact counterexample.

**References**


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