PERIODIC POINTS OF POSITIVELY EXPANSIVE MAPS

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Abstract. Stability of fixed points of a uniformly convergent sequence of open $e$-locally expansive maps of a compact connected locally connected metric space is studied by Hu and Rosen. Under the same assumption we prove a stability result stronger than theirs.

Hu and Rosen proved [4] a stability theorem for fixed points of a uniformly convergent sequence of open $e$-locally expansive maps of a compact connected locally connected metric space. We prove a stronger stability theorem by using the pseudo-orbit tracing property.

Let $(X, d)$ be a compact metric space, and let $f: X \to X$ be a continuous map. We assume that $f$ is onto. We say that $f$ is positively expansive if there exists a number $c > 0$ such that $x \neq y$ implies the existence of a positive integer $n$ such that $d(f^n(x), f^n(y)) > c$; $c$ is called a positively expansive constant for $f$. We say that $f$ is $e$-locally expansive if there is an $e > 0$ such that $0 < d(x, y) < e$ implies $d(f(x), f(y)) > d(x, y)$. We say that $f$ is an $e$-local expansion with skewness $\lambda$ if there exist $e > 0$ and $\lambda > 1$ such that $0 < d(x, y) < e$ implies $d(f(x), f(y)) > \lambda d(x, y)$. In [5] Reddy proved that if $f$ is $e$-locally expansive, then $f$ is positively expansive with constant $e/2$, and there is a compatible metric $\rho$ for $X$ such that $f$ is an $e_1$-local expansion with skewness $\lambda$ under $\rho$ for some constants $e_1 > 0$ and $\lambda > 1$. Rosenholtz showed in [7] that if $f$ is an open $e$-locally expansive and if $X$ is connected, then $f$ has a fixed point. Given $\delta > 0$, a sequence $\{x_j\}^n_{j=0}$ is called a $\delta$-pseudo-orbit ($\delta$-p.o.) of $f$ if $d(f(x_j), x_{j+1}) < \delta$ for $0 \leq j \leq n - 1$. Given $\eta > 0$, a sequence $\{x_j\}^n_{j=0}$ is said to be $\eta$-traced by a point $y$ in $X$ if $d(f^j(y), x_j) < \eta$ for $0 \leq j \leq n$. We say that $f$ has the pseudo-orbit tracing property (P.O.T.P.) if for each $\eta > 0$ there is $\delta > 0$ such that every $\delta$-p.o. of $f$ can be $\eta$-traced by some point in $X$. We remark that this property is independent of the metric for $X$. It is well known that if $f$ is an open $e$-local expansion, then $f$ has the P.O.T.P. (cf. Proposition 3.6 of [2]).

The following is proved.

Theorem. Let $(X, d)$ be a compact connected locally connected metric space, and let $f_i: X \to X$ be open $e$-locally expansive maps for $i = 0, 1, 2, \ldots$ such that the sequence
\{(f_n)_{n=1}^\infty\} converges uniformly to \(f_0\). Then for any \(\eta > 0\) there is a positive integer \(N\) such that, for each \(n \geq 1\) and each fixed point \(a_0\) of \(f_0^n\), there exists a sequence of fixed points \(a_i\) of \(f_n\) with \(d(a_i, a_0) < \eta\) for each \(i \geq N\).

First of all we prepare lemmas that we shall need.

**Lemma 1.** Let \((X, \rho)\) be a compact metric space, and let \(f: X \rightarrow X\) be an open \(\varepsilon_1\)-local expansion with skewness \(\lambda\). Then there is a positive number \(\delta_0 < \varepsilon_1/2\) such that \(x, y \in X\) and \(\rho(f(x), y) < \delta_0\) implies \(B_{\delta_0/\lambda}(x) \cap f^{-1}(y) \neq \emptyset\), where \(B_\alpha(x) = \{z \in X: \rho(x, z) < \alpha\}\).

**Proof.** See Lemma 1 of [3].

**Lemma 2.** Let \((X, \rho)\) be a compact connected metric space, and let \(f: X \rightarrow X\) be an open \(\varepsilon_1\)-local expansion with skewness \(\lambda\). Then per\((f)\), the set of all periodic points of \(f\), is dense in \(X\).

**Proof.** Let \(\delta_0\) be as in Lemma 1. It is enough to see that for each \(\nu > 0\) (\(\nu < \delta_0/2\)) there is a positive integer \(r\) such that \(f^r\) has a fixed point in \(B_\rho(x)\). For each number \(\nu > 0\), choose \(\delta > 0\) as in the definition of the P.O.T.P. Let \(\mathcal{U} = \{U_1, U_2, \ldots, U_n\}\) be a finite open cover of \(X\) so that the diameter of each \(U_k \in \mathcal{U}\) is less than \(\delta_0\). Then we can choose a positive integer \(r\) such that \(r\delta_0/\lambda^r < \delta\). Take and fix \(x_0 \in X\). Since \(X\) is connected, there is a finite sequence \(\{y_0 = x_0, y_1, \ldots, y_k, y_{k+1} = f^r(y_0)\}\) such that \(y_j, y_{j+1} \in U_j \in \mathcal{U}\) for \(0 \leq j \leq k (\leq n)\). By Lemma 1 there is \(y_k \in X\) with the property that \(f(y_k) = y_k\) and \(\rho(y_k, f^{-1}(x_0)) < \delta_0/\lambda\) (since \(\rho(y_k, f^{-1}(x_0)) < \delta_0\)). Similarly, since \(\rho(y_{k-1}, f(y_k)) < \delta_0\), there is \(y_{k-1}\) with the property that \(f(y_{k-1}) = y_{k-1}\) and \(\rho(y_{k-1}, y_k) < \delta_0/\lambda\). Continuing in this fashion, we can construct a finite sequence \(\{y_0 = x_0, y_1, \ldots, y_k, y_{k+1} = f^r(x_0)\}\). Next, since \(\rho(y_k, f^{-1}(x_0)) < \delta_0\), by Lemma 1 there is \(y_k \in X\) with the property that \(f(y_k) = y_k\) and \(\rho(y_k, f^{-1}(x_0)) < \delta_0/\lambda^2\). Similarly, since \(\rho(y_{k-1}, f(y_k)) < \delta_0/\lambda\), there is \(y_{k-1}\) in \(X\) with the property that \(f(y_{k-1}) = y_{k-1}\) and \(\rho(y_{k-1}, y_k) < \delta_0/\lambda^2\). In this manner we can find a finite sequence \(\{y_0 = x_0, y_1, \ldots, y_k, y_{k+1} = f^{r-1}(x_0)\}\). Inductively, for \(1 \leq l \leq r-1\), we can construct a finite sequence \(\{y_l = x_l, y_{l+1}, \ldots, y_k, y_{k+1} = f^{r-l}(x_0)\}\). Then

\[\rho(x_{r-l}, f(x_0)) \leq \rho(y_{r-l}^{-1}, y_{r-l}^{-1}) + \cdots + \rho(y_0^{-1}, f(x_0)) \leq n\delta_0/\lambda^{r-1} < \delta.\]

Hence, \(\{x_0, x_{r-1}, x_{r-2}, \ldots, x_1, x_0, x_{r-1}, \ldots\}\) is a \(\delta\)-p.o. of \(f\). Since \(f\) has the P.O.T.P. and \(f\) is positively expansive with constant \(\delta_0\), there is a tracing point \(p \in B_\rho(x_0)\) such that \(f^r(p) = p\) (cf. [1]).

**Lemma 3.** Let \((X, d)\) be a compact connected metric space, and let \(f: X \rightarrow X\) be open and \(\varepsilon\)-locally expansive. If \(E\) is a closed nonempty subset of \(X\) with \(f^{-1}(E) \subseteq E\), then \(E = X\).

**Proof.** There exist a compatible metric \(\rho\) for \(X\) and numbers \(\varepsilon_1 > 0\) and \(\lambda > 1\) such that \(f\) is an open \(\varepsilon_1\)-local expansion with skewness \(\lambda\) under \(\rho\). To arrive at the conclusion, it is enough to prove that \(\text{per}(f) \subseteq E\). Let \(p \in \text{per}(f)\) with \(f^r(p) = p\). Since \(X\) is compact, there are \(\varepsilon' > 0\) and \(\lambda' > 1\) such that \(f^r\) is an open \(\varepsilon'\)-local
expansion with skewness \( X \) under \( \rho \). Hence, we can find \( \delta_0 > 0 \) satisfying Lemma 1 for \( f' \). Let \( \mathcal{U}' = \{ U'_1, U'_2, \ldots, U'_r \} \) be a finite open cover of \( X \) so that the diameter of each \( U'_j \in \mathcal{U}' \) is less than \( \delta_0 \) under \( \rho \). Take \( y_0 \in E \). Since \( X \) is connected, there is a finite sequence \( \{ z_0 = y_0, z_1, \ldots, z_k, z_{k+1} = f'(p) = p \} \) such that \( z_j, z_{j+1} \in U'_j \) for \( 0 \leq j \leq k' \) (\( \leq n' \)). By the argument used in the proof of Lemma 2, for \( m > 0 \) we can construct a finite sequence \( \{ z^m_0 = y^m_0, z^m_1, \ldots, z^m_k, z^m_{k+1} = f'(p) = p \} \) with the property that \( y^m_m \in f^{-m}(y_0) \subset E \) and \( \rho(z^m_0, z^m_1) + \cdots + \rho(z^m_k, p) \leq m\delta_0/\lambda^m \). Hence, \( y^m_m \to p \) (\( m \to \infty \)), so \( \text{per}(f) \subseteq E \).

The following lemma is essentially due to Hu and Rosen [4].

**Lemma 4.** Let \((X, d)\) be a compact connected locally connected metric space, and let \( f_i: X \to X \) be open \( \epsilon \)-locally expansive maps for \( i = 0, 1, 2, \ldots \) such that the sequence \( \{ f_i \}_{i=1}^{\infty} \) converges uniformly to \( f_0 \). Then there is an integer \( N_1 > 0 \) such that \( i \geq N_1 \) implies \( \text{card}(f_i^{-1}(x)) = \text{card}(f_0^{-1}(y)) \) for all \( x, y \in X \).

**Proof.** According to Lemma 2 in [6], there is a finite open cover \( \{ W_\beta \} \) of \( X \) such that, for each \( \beta \) and for \( i = 0, 1, 2, \ldots \), \( \text{diam } W_\beta < \epsilon/2 \) and \( f_i \) maps every component of \( f_i^{-1}(W_\beta) \) homeomorphically onto \( W_\beta \). Let \( \alpha \) be a Lebesgue number for \( \{ W_\beta \} \). It is easy to check that if \( x, y \in X \) and \( d(f_i(x), y) < \alpha \), then \( B_\alpha(x) \cap f_i^{-1}(y) \) consists of a single point for each \( i \) by the \( \epsilon \)-local expansiveness of \( f_i \). Here \( B_\alpha(x) = \{ z \in X: d(x, z) < \alpha \} \). Since \( f_i \to f_0 \) uniformly, there is an \( N_1 \) such that \( i \geq N_1 \) implies \( d(f_i(x), f_0(x)) < \alpha \) for all \( x \in X \). Fix \( x_0 \in X \). Now suppose \( i \geq N_1 \) and \( z \in f_0^{-1}(x_0) \). Then \( d(f_i(x), x_0) < \alpha \). Thus, there is a unique \( x' \in X \) per \( z \) such that \( d(z, x') < \epsilon/2 \) and \( x' \in f_i^{-1}(x_0) \). Hence, \( \text{card}(f_i^{-1}(x_0)) \geq \text{card}(f_0^{-1}(x_0)) \). A similar argument shows that \( \text{card}(f_i^{-1}(x_0)) \leq \text{card}(f_0^{-1}(x_0)) \). By Property 4.5 in [4], we get the conclusion.

**Proof of Theorem.** Take \( \eta \) with \( 0 < \eta < \epsilon/6 \). Let \( \delta > 0 \) be as in the definition of the P.O.T.P. of \( f_0 \). Since \( f_i \to f_0 \) uniformly, there is an integer \( N \geq N_1 \) such that \( i \geq N \) implies \( d(f_i(x), f_0(x)) < \delta \) for all \( x \in X \). Here \( N_1 \) is an integer given by Lemma 4. Then for \( i \geq N \) there are unique continuous one-to-one (not necessarily onto) maps \( h_i: X \to X \) with \( h_i f_i = f_i h_i \), and \( d(h_i(x), x) < \eta \) for all \( x \in X \) (cf. [8, pp. 236–238]). By Lemma 4 we easily obtain that \( f_0^{-1} h_i(X) \subset h_i(X) \) for \( i \geq N \). Indeed, we may assume that there is an integer \( k \) such that, for all \( i \geq N \) and for all \( x \in X \), \( \text{card}(f_i^{-1}(x)) = \text{card}(f_0^{-1}(x)) = k \). Hence, for each \( x' \in h_i(X) \), there are exactly \( k \)-points \( \{ y_1, y_2, \ldots, y_k \} \) in \( X \) such that \( f_i(y_m) = h_i^{-1}(x') \) for \( 1 \leq m \leq k \) (since \( h_i \) is one-to-one). Obviously, \( f_0 h_i(y_m) = h_i f_i(y_m) = x' \) for \( 1 \leq m \leq k \). That is,

\[
\begin{align*}
  f_0^{-1}(x') = \{ h_i(y_1), h_i(y_2), \ldots, h_i(y_k) \} \subset h_i(X)
\end{align*}
\]

(since \( h_i \) is one-to-one and \( \text{card}(f_i^{-1}(x')) = k \)). Hence, by Lemma 3 we have \( h_i(X) = X \) for \( i \geq N \). Let \( a_0 \) be a fixed point of \( f_0^n \) and put \( a_i = h_i^{-1}(a_0) \). Then \( f_i^n h_i = h_i f_i^n \) implies \( f_i^n(a_i) = a_i \), and \( d(h_i(x), x) < \eta \) for all \( x \in X \) implies \( d(a_i, a_0) < \eta \) for all \( i \geq N \) and for all \( n \geq 1 \). □

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