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A METRIC ON HYPERSPACES DEFINED BY WHITNEY MAPS

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ABSTRACT. For a given continuum X a new metric on the hyperspace 2^X is defined, which is equivalent to the Hausdorff distance, but which has some other properties.

All spaces in this paper are assumed to be metric and all mappings are continuous. A continuum is a compact connected space. Given a continuum X with a metric d, we define the Hausdorff distance H between two nonempty closed subsets A and B by

$$H(A, B) = \max\left\{ \sup_{a \in A} \inf_{b \in B} d(a, b),\ \sup_{b \in B} \inf_{a \in A} d(a, b) \right\}$$

(see [**1**, (0.4), p. 3]). The symbol 2^X denotes the hyperspace of all nonempty closed subsets of a continuum X with the Vietoris topology (see [**1**, (0.11), p. 9] for the definition) or, equivalently (see [**1**, (0.13), p. 10]) with the topology determined by the Hausdorff distance.

A mapping μ: $2^X \to [0, \infty)$ is called a Whitney map (see [**1**, (0.50), p. 24]) if it satisfies the conditions:

(1) for every $x \in X$, $\mu(\{x\}) = 0$; and

(2) for every $A, B \in 2^X$ with $A \subset B$ and $A \neq B$, $\mu(A) < \mu(B)$.

We consider special Whitney maps, namely ones satisfying an additional condition:

(3) for every $A, B \in 2^X$ with $A \subset B$ and for every $C \in 2^X$,

$$\mu(B \cup C) - \mu(A \cup C) \leqslant \mu(B) - \mu(A).$$

Such mappings do exist for every continuum X (see Proposition 1 below).

Given a sequence of sets $\{A_n\}_{n=1}^{\infty}$ we denote by Ls A_n the upper limit of the sequence in the sense of [**1**, (0.5), p. 4], and by Lim A_n the limit of the sequence in the sense of [**1**, (0.5), p. 4] or, equivalently (see [**1**, (0.7), p. 4]), in the sense of the Hausdorff distance.

In the present paper a new metric on the hyperspace of a continuum is defined, which is equivalent to the Hausdorff distance, but which has some other properties.

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We start with

Proposition 1. *For every continuum X there are Whitney maps μ and μ' such that μ satisfies, while μ' does not satisfy, condition* (3).

Really, the reader can verify that a Whitney map μ defined in [**1**, (0.50.2), p. 26] has property (3). On the other hand, let $x, y, z \in X$ be any distinct points and put $f(\{x\}) = f(\{y\}) = f(\{z\}) = 0$, $f(\{x, y\}) = f(\{x, z\}) = f(\{y, z\}) = 1$, and $f(\{x, y, z\}) = 3$. Then f satisfies (1) and (2) for the space $\{x, y, z\}$ and therefore it can be extended to a Whitney map μ' on 2^X (see [**2**, Corollary 3.4, p. 468] and observe that the assumption of connectedness of spaces is not used in the proof). However, putting $A = \{x\}$, $B = \{x, y\}$, and $C = \{z\}$, we can see that f (and hence μ') does not satisfy (3).

Definition 2. Let X be a continuum and let μ be a Whitney map satisfying (3). Define, for every $P, Q \in 2^X$,

$$D_\mu(P, Q) = \max\{\mu(P \cup Q) - \mu(P), \mu(P \cup Q) - \mu(Q)\}.$$

Proposition 3. *D_μ defined above is a metric on 2^X.*

Proof. The condition $D_\mu(P, Q) = 0$ if and only if $P = Q$ is a consequence of (2); the symmetry of D_μ is obvious from the definition. We show the triangle condition. Let $P, Q, R \in 2^X$. We can assume without loss of generality that $\mu(P) \leqslant \mu(R)$. Then we have to show

$$\mu(P \cup Q) - \min\{\mu(P), \mu(Q)\} + \mu(Q \cup R) - \min\{\mu(Q), \mu(R)\} \geqslant \mu(P \cup R) - \mu(P).$$

It is enough to show

$$\mu(P \cup Q) - \mu(P) + \mu(Q \cup R) - \mu(Q) - \mu(P \cup R) + \mu(P) \geqslant 0,$$

but using (3) for $A = Q$, $B = P \cup Q$, and $C = R$ we see that the left member of the inequality is greater than or equal to

$$\mu(P \cup Q \cup R) - \mu(Q \cup R) + \mu(Q \cup R) - \mu(P \cup R)$$

and, therefore, is nonnegative.

Proposition 4. *For any Whitney map μ satisfying* (3) *the metric D_μ is equivalent to the Hausdorff distance H.*

Proof. Let a set $A \in 2^X$ be given and assume a sequence $\{A_n\}_{n=1}^{\infty}$ tends to A with respect to the Hausdorff distance, i.e., $H(A_n, A) \to 0$. Then $H(A_n \cup A, A) \to 0$, and by continuity of μ we have $\mu(A_n \cup A) \to \mu(A)$ and $\mu(A_n) \to \mu(A)$. Thus,

$$\max\{\mu(A_n \cup A) - \mu(A), \mu(A_n \cup A) - \mu(A_n)\} \to 0,$$

i.e., the sequence $\{A_n\}_{n=1}^{\infty}$ tends to the set A with respect to the metric D_μ.

On the other hand assume $\{A_n\}_{n=1}^{\infty}$ tends to A with respect to the metric D_μ, i.e.,

(4) $\mu(A_n \cup A) - \mu(A) \to 0$ and

(5) $\mu(A_n \cup A) - \mu(A_n) \to 0$.

We show that

(6) $\mathrm{Lim}(A_n \cup A) = A$.

Assume, on the contrary, that there is a subsequence $\{A_{n_i}\}_{i=1}^{\infty}$ with $\mathrm{Lim}(A_{n_i} \cup A) = B \neq A$. Then $A \subset B$ and (2) imply $\mu(A) < \mu(B)$, a contradiction to (4).

Note that (6) implies

(7) $\mathrm{Ls} A_n \subset A$.

Now suppose there exists a subsequence $\{A_{n_j}\}_{j=1}^{\infty}$ with $\mathrm{Lim}\, A_{n_j} = C \neq A$. By (7) we have $C \subset A$ and, therefore, by (2), $\mu(C) < \mu(A)$. Then (6) implies a contradiction to (5). So we have proved $\mathrm{Lim}\, A_n = A$, i.e., $\{A_n\}_{n=1}^{\infty}$ tends to A with respect to the Hausdorff distance.

Now we show some facts concerning the metric D_μ. Some of them are obvious and their proofs are omitted.

Let X be a fixed continuum and let μ be a Whitney map satisfying (3).

FACT 5. *Consider 2^X as a metric space with the metric D_μ, and let $\mathscr{A} \subset 2^X$ be an ordered arc. Then $\mu|_{\mathscr{A}}: \mathscr{A} \to [0, \infty)$ is an isometry.*

FACT 6. *Let $x \in A \in 2^X$. Then $D_\mu(A, \{x\}) = \mu(A)$. In other words, the distance between a set and any point in the set does not depend on the choice of the point.*

FACT 7. *Let $\mathscr{A}$ be an ordered arc contained in 2^X and let $P \in 2^X$. Denote by A_0 either the only set in $\mathscr{A}$ satisfying $\mu(A_0) = \mu(P)$ if such a set does exist, or $\bigcap \mathscr{A}$ if $\mu(P) < \mu(A)$ for each $A \in \mathscr{A}$, or $\bigcup \mathscr{A}$ if $\mu(P) > \mu(A)$ for each $A \in \mathscr{A}$. Then* $\inf\{D_\mu(A, P): A \in \mathscr{A}\} = D_\mu(A_0, P)$.

PROOF. Take a set $A \in \mathscr{A}$. We have to show $D_\mu(A_0, P) \leqslant D_\mu(A, P)$. Consider two cases:

Case 1. $A_0 \subset A$. Then

$$D_\mu(A, P) = \mu(A \cup P) - \mu(P) \geqslant \mu(A_0 \cup P) - \mu(P) = D_\mu(A_0, P).$$

Case 2. $A \subset A_0$. Then by (3) we have

$$D_\mu(A, P) = \mu(A \cup P) - \mu(A) \geqslant \mu(A_0 \cup P) - \mu(A_0) = D_\mu(A_0, P).$$

This completes the proof.

FACT 8. *Let D be any metric on 2^X equivalent to the Hausdorff metric. Then the continuity of a Whitney map μ means*

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall A, B \in 2^X: D(A, B) < \delta \Rightarrow |\mu(A) - \mu(B)| < \varepsilon.$$

If we replace D by D_μ we can put $\delta = \varepsilon$.

PROOF. We have to show $D_\mu(A, B) < \varepsilon$ implies $|\mu(A) - \mu(B)| < \varepsilon$. Assume $\mu(A) \geqslant \mu(B)$. Then

$$\varepsilon > D_\mu(A, B) = \mu(A \cup B) - \mu(B) \geqslant \mu(A) - \mu(B),$$

and we are done.

To end the paper we ask some questions connected with condition (3). We say that two Whitney maps μ_1 and μ_2 are equivalent if for every t there exist t' and t'' such that $\mu_1^{-1}(t)$ is homeomorphic to $\mu_2^{-1}(t')$ and $\mu_2^{-1}(t)$ is homeomorphic to $\mu_1^{-1}(t'')$.

Question 9. Given any Whitney map μ_1 is there a Whitney map μ_2 which is equivalent to μ_1 and satisfies (3)?

Question 10. Given any continuum X and any Whitney map μ: $2^X \to [0, \mu(X)]$ does there exist a homeomorphism h from $[0, \mu(X)]$ into $[0, \infty)$ such that $h \circ \mu$ is a Whitney map satisfying (3)?

References

1. Sam B. Nadler, Jr., *Hyperspaces of sets*, Dekker, New York, 1978.
2. L. E. Ward, Jr., *Extending Whitney maps*, Pacific J. Math. **93** (1981), 465–469.

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