STRUCTURAL INSTABILITY OF $e^z$

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Abstract. The entire function $e^z$ has a Julia set equal to the whole plane. We show that there are complex $\lambda$'s near 1 such that $\lambda e^z$ has an attracting periodic orbit. Hence $e^z$ is not structurally stable.

The study of the dynamics of complex analytic maps goes back to Fatou and Julia in the 1920s. Recently, there has been a rebirth of interest in this subject, due mainly to the interesting new results of Mandelbrot [Ma], Douady-Hubbard [DH], and Sullivan [S]. Most of this recent work has dealt with polynomials or rational maps, rather than with entire maps. The reason for this is that rational maps may be regarded as smooth maps of the Riemann sphere, whereas, because of the essential singularity at infinity, entire maps cannot. Thus, compactness is lost and, with it, many of the global theorems in the subject.

Nevertheless, entire maps like $e^z$ share many of the dynamical features of rational maps. However, as was shown in [DK], the exponential map has several additional features that are not found in the polynomial case. For example, Misiurewicz [M] has shown that the Julia set of $e^z$ is the entire plane, something that cannot occur for polynomials. Recall that the Julia set of a map is the set of points at which the family of iterates of the map fails to be a normal family of functions, and that periodic points are dense in the Julia set [F]. In particular, periodic points for $e^z$ are dense in the plane.

This raises the question of the structural stability of $e^z$. One may check easily that if an entire function is topologically conjugate to $e^z$, then it is affinely equivalent to $\lambda e^z$ for some $\lambda \in \mathbb{C}$. Hence, structural stability here means within the class of maps $\lambda e^z$. In this limited class of maps, it would seem that there is a good chance for stability. Nevertheless, our main result is that this is not so.

Theorem. $e^z$ is not structurally stable.

We remark that this result is a crucial step in the recent proof of Ghys, Goldberg, and Sullivan [GGS] that $e^z$ is recurrent (in the sense that there are no positive measure cross sections to the grand orbits of $e^z$). The natural question of the ergodicity of $e^z$ remains open.
To show that $e^z$ is not structurally stable, we prove that there exist $\lambda$-values arbitrarily close to $1$ for which $\lambda e^z$ has an attracting periodic point. In fact, we show that these periodic points can have arbitrarily large periods as $\lambda$ approaches $1$.

Throughout this paper we assume that $\lambda = e^{i\theta}$, with $0 \leq \theta < \pi/2$, and write $f_\theta(z) = \lambda e^z$. Let $S$ be the strip determined by $\text{Re} z \geq 2$, $0 \leq \text{Im} z \leq \pi$. Let $z_0 = x_0 + iy_0$. We write $z_1 = z_1(\theta) = x_1 + iy_1 = f_\theta(z_0)$.

**Lemma 1.** There exists $\theta_1$, $0 < \theta_1 < \pi/2$, such that, if $0 < \theta < \theta_1$ and both $z_0, z_1 \in S$, then

(i) $x_1(\theta) > 2x_0 + 1$,
(ii) $y_1(\theta) > 2y_0$.

**Proof.** First let $\theta = 0$. Then $e^{x_0} \sin y_0 \leq \pi$, so that $\sin y_0 \leq \pi e^{-2}$. Hence $\cos y_0 > 0.8$. Therefore, $x_1 = e^{x_0} \cos y_0 > 0.8 e^{x_0} > 2x_0 + 1.5$. Also, $y_1 = e^{x_0} \sin y_0 > e^{x_0} y_0/2 > y_0 e^{2}/2 > 3y_0$. So (i) and (ii) certainly hold for $\theta = 0$.

Now, for $\theta \neq 0$, we note first that $f_\theta(z) = e^{i\theta} f_0(z)$. If $z, f_\theta(z) \in S$, then it follows that $f_\theta(z) \in S$ as well. Hence it suffices to find $\theta_1$ such that, if $\theta < \theta_1$, then

$$0 < \text{Re}(w - e^{i\theta}w) < 1/2,$$
$$\text{Im}(e^{i\theta}w) > \text{Im}(w)$$

for all $w \in S$ such that $e^{i\theta} w \in S$. (2) clearly holds if $\theta_1 < \pi/2$. For (i), we observe that if $e^{i\theta} w \in S$, then $\text{Re} w < \pi/\sin \theta$. Therefore,

$$0 < \text{Re}(w - e^{i\theta}w) < (\text{Re} w)(1 - \cos \theta) + \pi \sin \theta$$
$$< \pi(1 - \cos \theta)/\sin \theta + \pi \sin \theta.$$

Since the right side approaches $0$ as $\theta \to 0$, we may choose $\theta_1$ small enough so that (i) holds for $\theta < \theta_1$.

We remark that there exists $\theta_2 > 0$ such that $f_\theta^2(0) \in S$ for $0 \leq \theta \leq \theta_2$. From now on we assume that $\theta < \min(\theta_1, \theta_2)$.

Define $G_n(\theta) = f_\theta^n(0)$. $G_1(\theta)$ traces out the unit circle, while $G_2(\theta)$ gives a cardioid-like curve, part of which meets $S$. Let $\gamma_2(\theta)$ denote the piece of $G_2(\theta)$ in $S$. Note that $\gamma_2(\theta)$ meets $y = 0$ at $f_\theta^2(0) = e$. Let $\gamma_n(\theta)$ denote the connected component of $S \cap G_n(\theta)$ which contains $f_\theta^n(0)$. For $n$ sufficiently large (numerically, $n \geq 3$), $\gamma_n(\theta)$ connects $y = 0$ to $y = \pi$ in $S$. More precisely, for $n \geq 3$, there exists $\theta_n$ such that $\text{Im}(\gamma_n(\theta_n)) = \pi \text{ Re}(\gamma_n(\theta_n)) \geq 2$, and, for all $\theta$ with $0 < \theta < \theta_n$, $\gamma_n(\theta) \in S$. See Figure 1. This can be seen by applying Lemma 1 repeatedly to $f_\theta(\gamma_2(\theta))$. If $\theta > 0$, there exists $n = n(\theta)$ such that $f_\theta^n(\gamma_2(\theta)) \notin S$.

Clearly, $\exp(\gamma_n(\theta))$ is a curve in the upper half-plane which connects the positive real axis to the negative real axis. Since $0 < \theta < \pi/2$, the curve $f_\theta(\gamma_n(\theta)) = e^{i\theta} \exp(\gamma_n(\theta))$ also meets the negative real axis. When $\theta = \theta_n$, the image $f_\theta(\gamma_n(\theta))$ is negative and real. Denote this point by $z_{n+1}$, so that $z_{n+1} = f_\theta^{n+1}(0)$. Also let $z_j = f_\theta^j(0) = x_j + iy_j$ for $0 \leq j \leq n + 1$. Note that $z_0 = 0$, $|z_1| = 1$, and $z_j \in S$ for $2 \leq j \leq n$.

**Lemma 2.** $\exp(x_n) \geq 2 + \sum_{j=1}^n (x_j + 1)$. 

PROOF. We have $x_2 \geq x_1 + 1$. Moreover, by Lemma 1, for $2 \leq j \leq n - 1$, we have $x_{j+1} \geq 2x_j + 1$. Hence,

$$2 + \sum_{j=1}^{n} (x_j + 1) \leq \sum_{j=2}^{n-1} (x_{j+1} - x_j) + x_2 + x_n + 3 = 2x_n + 3 < e^{x_n},$$

since $x_n \geq 2$.

We now construct a disk about 0 which is contracted inside itself by $f_0^{n+2}$. Let $r_{n+1} = 1$ and define $r_k = r_{k+1}/e^{x_k}$ for $0 \leq k \leq n$. Note that $r_j < 1$ for $j \leq n$ and $r_0 = (e^{n+1} \prod_{j=0}^{n} e^{x_j})^{-1}$. Let $B_j$ be the disk of radius $r_j$ about $z_j$.

**Proposition 3.** If $z \in B_0$, then $f_0^j(z) \in B_j$ for $j \leq n + 1$. Moreover,

$$\left| \left( f_0^{n+1} \right)'(z) \right| \leq e^{n+1} \prod_{j=0}^{n} e^{x_j}.$$

**Proof.** Suppose $|z - z_j| < r_j$. Let $M_j = \sup |f_0'(z)|$. Then, for $j \leq n$,

$$|f_0(z) - z_{j+1}| \leq M_j r_j \leq |\lambda e^{x_j} r_j| r_j \leq e^{x_j} r_j \cdot r_j < e^{x_j} r_j = r_{j+1},$$

since $r_j < 1$. Consequently, $f_0$ maps $B_j$ strictly inside $B_{j+1}$, and we have $|f_0^j(z)| \leq e^{x_j+1}$. The result follows immediately.

**Theorem.** $f_0$ has an attracting periodic point of period $n + 2$ in $B_0$.

**Proof.** By the preceding proposition, $f_0^{n+1}(B_0) \subset B_{n+1}$. We now show that $f_0(B_{n+1}) \subset B_0$. Let $z \in B_{n+1}$. Then $\text{Re } z \leq x_{n+1} + 1 = -e^{x_n} + 1$. Applying Lemma 2, we have

$$|f_0(z)| \leq \exp(-e^{x_n} + 1) \leq e \exp \left( -\sum_{j=1}^{n} (x_j + 1) - 2 \right)$$

$$= e^{-1} \left( \prod_{j=0}^{n} e^{x_j} \right)^{-1} e^{-n} = r_0,$$

as required. Hence, there exists a periodic point $w$ such that $f_0^j(w) \in B_j$ for $0 \leq j \leq n + 1$ and $f_0^{n+2}(w) = w$. 

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**Figure 1.** The curves $G_n(\theta)$ for $n = 1, 2, 3$. 

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<th>$G_1$</th>
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<td>$y = 0$</td>
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Finally, observe that $|f'_n(z)| < r_0$ if $z \in B_{n+1}$. Combined with the results of Proposition 3, this yields $|(f^m_{m+2})'(w)| < 1$. This completes the proof.

**Corollary.** $e^z$ is not structurally stable.

**Remark 1.** There are no critical points for the map $\lambda e^z$, and so, in particular, there are no critical values in the immediate attractive basin of $w$. This cannot happen for rational maps, where basins of attraction must contain the forward orbit of a critical value. The analogy here is that the omitted value 0 plays the role of the critical value.

**Remark 2.** The orbit of the omitted value is a topological invariant; if $g(z)$ is conjugate to $e^z$, $g(z)$ must have a unique omitted value which is mapped to 0 by the conjugacy.

**Remark 3.** There are other, dramatically different, possibilities for the orbit of 0 under $\lambda e^z$; for example, 0 may be eventually periodic. We sketch a proof. Consider the family of functions of $\lambda$ given by $G_n(\lambda) = f^n_\lambda(0)$. Using the above, we easily see that $G_n$ is not a normal family in any neighborhood of 1. Hence, by Montel's Theorem, there exist $\lambda$, arbitrarily close to 1, and $n$, such that $G_n(\lambda) = 2k\pi i$ for $k \neq 0$. But then

$$F^n_\lambda(\lambda) = f^n_\lambda f_\lambda(0) = f_\lambda(2k\pi i) = \lambda,$$

so that $\lambda$ is a periodic point. In fact, $\lambda$ is a repelling periodic point, and one can show that the Julia set of $f_\lambda$ is the entire plane. This follows from the classification theorem of Sullivan [S], which extends with some modifications to the case of $\lambda e^z$. See [GK and D].

**Remark 4.** It is well known that the Julia set of a complex analytic map is either $\mathbb{C}$ or nowhere dense. Since the basins of attraction of attracting periodic points are never in the Julia set, it follows that the Julia set of $\lambda e^z$ changes radically in every neighborhood of 1. The structure of these Julia sets is not yet well understood.

**References**


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