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ON TRIVIAL INTERSECTION OF CYCLIC SYLOW SUBGROUPS

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ABSTRACT. The classification of finite simple groups is used to prove that a cyclic Sylow subgroup of a finite simple group must be a trivial intersection set. Applications to character theory, and a necessary and sufficient condition for a cyclic Sylow subgroup of an arbitrary finite group to be a trivial intersection set, are obtained as corollaries.

1. Introduction. The main object of this paper is to prove, with the aid of the classification of all finite simple groups, the following[1]

THEOREM 1. *If P is a cyclic Sylow p-subgroup of a finite simple group G, then P is a trivial intersection (T.I.) set in G.*

The theorem is nontrivial as it applies to a cyclic Sylow subgroup which is not of prime order. Such subgroups occur frequently in finite simple groups of Lie type.

Before we prove Theorem 1, we will state some corollaries and give some examples. Throughout the paper, p will be a fixed rational prime. If P is a finite cyclic p-group, $\tilde{P}$ denotes the unique subgroup of P of order p. By "simple", we mean, of course, "non-Abelian simple".

COROLLARY 1. *Let G be a finite group with a cyclic Sylow p-subgroup P, and let $U = O_{p'}(G)$. Then P is a T.I. set in G if and only if $C_U(P) = C_U(\tilde{P})$.*

COROLLARY 2. *Let G be a finite simple group with a cyclic Sylow p-subgroup P. If χ is an irreducible character of G, then either $|P| \mid \chi(1)$ or $(p, \chi(1)) = 1$.*

COROLLARY 3. *Let G be a finite simple group with a cyclic Sylow p-subgroup P. If $\chi \neq 1_G$ is an irreducible character of G, then either $G \approx \mathrm{PSL}_2(p)$ or $\chi(1) \geq |P| - 2$.*

Here are some instances of a group G with a cyclic Sylow subgroup P which is *not* a T.I. set. The solvable group G of [**7**, VII.11.1] is one such example.

EXAMPLE 1 (DUE TO W. FEIT). Let H be any finite group with a cyclic Sylow p-subgroup P such that $|P| > p$. The exceptional characters χ in the principal p-block of H are in bijection with a set of representatives $\{\lambda\}$ of $N_G(P)/C_G(P)$-orbits of $\mathrm{Irr}(P) - \{1\}$, such that $\chi = \chi_\lambda \leftrightarrow \lambda$ if and only if

$$\chi_{\lambda_P} = m\rho_P + \varepsilon \sum_{g \in N_G(P)/C_G(P)} \lambda^g$$

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[1] I learned, just prior to submitting this paper, that Theorem 1 has also been recently obtained, independently, by G. Michler (not yet published), as part of his broader results on the possible defect groups that occur in finite simple groups.

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where integer $m \geq 0$ and sign $\varepsilon = \pm 1$ are constant for all exceptional characters χ_λ in the principal block, and ρ_P is the regular representation of P [**7**, VII.2.17]. If $m = 0$, fix $\lambda \in \mathrm{Irr}(P) - \{1\}$ such that $\tilde{P} \leq \ker \lambda$. If $m > 0$, fix λ as a faithful linear character of P. Then

$$(1_P, \chi_{\lambda_P})_P = m < \left\{ \begin{array}{ll} |P\colon \tilde{P}|m & (\text{if } m > 0) \\ |N_G(P)\colon C_G(P)| & (\text{if } m = 0) \end{array} \right\} = (1_{\tilde{P}}, \chi_{\lambda_{\tilde{P}}})_{\tilde{P}}.$$

Let q be any prime with $q \nmid |H|$. Then there exists a splitting field $\mathrm{GF}(q^r)$ for H, and a module V for H over $\mathrm{GF}(q^r)$ which lifts to a module in characteristic zero which yields the character χ_λ [**7**, III.3.3]. Then V is an elementary Abelian q-group such that $C_V(P) < C_V(\tilde{P})$. Let G be the semidirect product VH. Then P is not a T.I. set in G by (the easy direction of) Corollary 1.

EXAMPLE 2. Let H be any finite group with a cyclic Sylow p-subgroup P such that $|P| > p$. Let H act (not necessarily faithfully) as a group of permutations on a set S of m letters, such that P acts nontrivially. Let J be any nontrivial p'-group and let V be the direct product of m copies of J. Let H act on V by permutation of coordinates. Since the nontrivial orbits of P on S break into properly smaller orbits for $\tilde{P}$, it is easily seen that $C_V(P) < C_V(\tilde{P})$. So if G is the semidirect product VH, then P is not a T.I. set in G.

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2. Proof of Theorem 1. The following result is both well known and elementary:

PROPOSITION (2.1). *Let P be a cyclic Sylow p-subgroup of a finite group G. If P is a T.I. set in G, then $C_H(P) = C_H(\tilde{P})$ for all subgroups H of G. Furthermore, P is a T.I. set in G if and only if $C_G(P) = C_G(\tilde{P})$.*

PROOF. Assume that P is a T.I. set in G. For any $H \leq G$, it is clear that $C_H(P) \leq C_H(\tilde{P})$. If $h \in C_H(\tilde{P})$, then $P^h \cap P \geq \tilde{P}$ implies that $h \in N_G(P)$, by assumption. Then $h_{p'}$ (the p'-part of h) normalizes P and centralizes $\tilde{P}$, hence must also centralize P. Also, $h_p \in P$. So $h = h_p h_{p'} \in C_H(P)$, and the first statement is proved.

To obtain the second statement, we may assume that $C_G(P) = C_G(\tilde{P})$. If $\tilde{P} \leq P^g \cap P$ for some $g \in G$ then $P^g P \leq C_G(\tilde{P}) = C_G(P)$, which has a unique Sylow p-subgroup P. Thus $P^g = P$ and P is a T.I. set.

PROOF OF THEOREM 1. Assume that G is a finite simple group with a cyclic Sylow p-subgroup P of order p^n. By Burnside's transfer theorem, $p > 2$.

If G is an alternating group, suppose that $g \in P$ has order p^2. Then g^p is a product of at least p disjoint cycles of length p, and hence is not conjugate in G to any power of a single p-cycle. So G has more than one conjugacy class of subgroups of order p. This contradicts the assumption that P is cyclic, and hence implies that $|P| = p$, so that P is clearly a T.I. set.

Suppose that G is a sporadic group. Then the list of normalizers of subgroups of prime order in G [**9**, pp. 40–70] shows that G has no cyclic Sylow subgroups apart from those of prime order.

By the classification of the finite simple groups (see [**8**, Chapter 2]), we may assume that G is of Lie type (nontwisted or twisted). Let $C = C_G(P)$, $\tilde{C} = C_G(\tilde{P})$. It suffices to prove $C = \tilde{C}$, by Proposition 2.1.

Let G, as a (possibly twisted) Chevalley group, be defined over the field $\mathrm{GF}(q)$, q a prime power. (When G is twisted, we mean here by "q" what Carter [**3**, p. 251] means by "q^2", "q^3", "2^{2m+1}" or "3^{2m+1}".) Suppose first that $p \mid q$. Then if G is nontwisted, P has distinct root subgroups X_s for each positive root s, with $X_s \approx (\mathrm{GF}(q), +)$, the additive group of $\mathrm{GF}(q)$ [**3**, 5.3.3]. Hence, P cyclic implies that $q = p$ and G has rank 1. Therefore, $G = A_1(p) = \mathrm{PSL}_2(p)$ and $|P| = p$, hence P is a T.I. set. If G is twisted, then P has distinct subgroups X_S^1 for each equivalence class S of positive roots (as in [**3**, Chapter 13]). Moreover, since $p > 2$, we have $G \not\approx {}^2B_2(2^{2m+1})$, ${}^2F_4(2^{2m+1})$ or ${}^2F_4(2)'$. Then $q > p$ and S has the structure of a fundamental system of a root system of type A_1, $A_1 \times A_1$ (both roots of the same length), $A_1 \times A_1 \times A_1$, A_2 or G_2 [**3**, 13.6.3]. If some S has type $A_1 \times A_1$ or $A_1 \times A_1 \times A_1$, then [**3**, 13.6.3, 13.6.4] implies that $X_S^1 \approx (\mathrm{GF}(q), +)$, an elementary Abelian group of order $q > p$, which contradicts our hypothesis on P. If some S has type G_2, then by [**3**, 13.6.4], the elements $x_S(0, u, v)$ of X_S^1 form an elementary Abelian group of order q^2, again a contradiction. If some S has type A_2, then X_S^1 has order $q^{3/2}$ and exponent p, but cyclic P has no such subgroup. Thus every class S of positive roots must be of type A_1, which contradicts the fact that the underlying symmetry of a Dynkin diagram used to define the twisted group G indeed has nontrivial orbits.

So we may assume that G is defined over $\mathrm{GF}(q)$ with $p \nmid q$. Let $\hat{G}$ be the universal Chevalley (or twisted Chevalley) group defined over $\mathrm{GF}(q)$ such that $\hat{G}/Z(\hat{G}) \approx G$. (If $G \approx {}^2F_4(2)'$, we may either handle this case separately or replace G by ${}^2F_4(2)$, in which G has index 2, as it suffices to show P is a T.I. set in ${}^2F_4(2)$.) Then there exists a simply connected, simple linear algebraic group G^*, defined over the algebraic closure of $\mathrm{GF}(q)$, and an endomorphism σ of G^* onto G^*, such that $\hat{G}$ equals G^*_σ, the fixed-point set of σ [**10**, 11.6, 12.8].

Let x generate P, and let $y = x^{p^{n-1}}$, so that $\tilde{P} = \langle y \rangle$. Let $Z = Z(\hat{G}) = O_p(Z) \times O_{p'}(Z)$. Let π be the natural projection of $\hat{G}$ onto G. Let $\hat{x}$ be a p-element in $\hat{G}$ such that $\pi(\hat{x}) = x$. Let $\hat{y} = \hat{x}^{p^{n-1}}$, so that $\pi(\hat{y}) = y$.

Let $T = \pi^{-1}(\tilde{C})$. Then $T = \{g \in \hat{G} | g^{-1}\hat{y}g = \hat{y}z, \text{ some } z \in Z\}$ and $C_{\hat{G}}(\hat{y}) \Delta T$. Now, $\hat{y}$ a p-element and $g^{-1}\hat{y}g = \hat{y}z$ imply that $z \in O_p(Z)$. Since $g^{-m}\hat{y}g^m = \hat{y}z^m$ for all integers m, it follows that $g^{p^b} \in C_{\hat{G}}(\hat{y})$ for some b. Thus, $T/C_{\hat{G}}(\hat{y})$ is a p-group. But $C_{\hat{G}}(\hat{y}) \geq \langle \hat{x}, O_p(Z) \rangle$, which is a Sylow p-subgroup of $\hat{G}$. It follows that $\pi^{-1}(\tilde{C}) = C_{\hat{G}}(\hat{y})$. Thus $\tilde{C} = C_{\hat{G}}(\hat{y})/Z$, and by a similar argument, $C = C_{\hat{G}}(\hat{x})/Z$. So it suffices to prove that $C_{\hat{G}}(\hat{x}) = C_{\hat{G}}(\hat{y})$.

A p'-automorphism of P must act faithfully on $\tilde{P}$. Hence, $N_{\tilde{C}}(P) = C_{\tilde{C}}(P)$. So Burnside's transfer theorem implies that $\tilde{C}$ has a normal p-complement, say H. Let $J = \pi^{-1}(H)$, so that $J/Z \approx H$. Thus $J/O_p(Z)$ is a p'-group, and so $J = O_p(Z) \times V$

for some p'-subgroup V, by the Schur-Zassenhaus Theorem. Thus,

$$C_{\hat{G}}(\hat{y}) \text{ has a normal } p\text{-complement } V. \tag{2.2}$$

Since $C_{\hat{G}}(\hat{y})/V \approx \langle \hat{x}, O_p(Z)\rangle$, which is Abelian, it follows that

$$V \geq C_{\hat{G}}(\hat{y})'. \tag{2.3}$$

Now $p\nmid q$ implies that $\hat{y}$ is a semisimple element of $\hat{G}$ (and of G^*), so by [**10**, 8.5], $C_{G^*}(\hat{y})$ is connected. Furthermore, $C_{G^*}(\hat{y})$ is a σ-stable reductive subgroup of maximal rank in G^*, and $C_{\hat{G}}(\hat{y}) = C_{G^*}(\hat{y})_\sigma$ [**4**, §1]. Also, $C_{G^*}(\hat{y}) = MS$, where M is semisimple, S is a torus central in $C_{G^*}(\hat{y})$, and both M and S are σ-stable [**4**, §3]. Moreover, by [**5**, §1],

$$|C_{\hat{G}}(\hat{y})| = |M_\sigma|\,|S_\sigma|. \tag{2.4}$$

Now by [**10**, 11.6, 11.19, 12.6], M_σ has the following structure: M_σ contains the central product of subgroups M_i, $1 \leq i \leq m$, where $|M_\sigma| = \prod_i |M_i|$ and each M_i has a normal series $Z_i \leq R_i \leq M_i$ such that $Z_i \leq Z(M_i)$, M_i/R_i is Abelian, and $R_i/Z_i := H_i$ is either a simple group of Lie type (possibly twisted) or is one of the eight finite adjoint groups which happen to be not simple (namely, $A_1(2) \approx \mathrm{PSL}_2(2)$, $A_1(3) \approx \mathrm{PSL}_2(3)$, $B_2(2)$, $G_2(2)$, ${}^2A_2(2^2) \approx \mathrm{PSU}_3(4)$, ${}^2B_2(2)$, ${}^2G_2(3)$, ${}^2F_4(2)$ [**3**, 11.1.2, 14.4.1]). The characteristic of H_i must divide q. Furthermore, if U_i is the universal Chevalley group which covers H_i, then $|M_i| \mid |U_i|$. It follows from [**8**, Tables 2.4 and 4.1] that

$$\text{If } r \text{ is any prime divisor of } |M_i|, \text{ then } r \mid |H_i|. \tag{2.5}$$

Since H_i' is a section of $C_{\hat{G}}(\hat{y})'$, it follows from (2.2) and (2.3) that H_i' is a p'-group. So if H_i is simple or if H_i/H_i' is a 2-group, then (2.5) implies that M_i is a p'-group.

Suppose that $p \mid |H_i|$ for some i, say $i = 1$. Then H_1 is one of the eight groups above, with $p \mid |H_1 : H_1'|$. It follows that $H_1 \approx A_1(3)$, ${}^2A_2(2^2)$ or ${}^2G_2(3)$ and $p = 3$ [**3**, pp. 176, 268]. But $\mathrm{SL}_3(4)$ has no element of order 9, hence a Sylow 3-subgroup of $\mathrm{PSU}_3(4)$ is elementary Abelian of order 9. So $H_1 \not\approx {}^2A_2(2^2)$, as P is cyclic. If $H_1 \approx {}^2G_2(3)$, then $3 \mid |{}^2G_2(3)'|$ [**3**, p. 268], which contradicts the fact that H_1' is a p'-group. Thus, $H_1 \approx A_1(3)$. But this forces 3 (the characteristic) to divide q, which contradicts $p\nmid q$.

It follows that H_i is a p'-group for $1 \leq i \leq m$. Then M_σ is a p'-group by (2.5). So (2.4) implies that $|C_{\hat{G}}(\hat{y})|_p \mid |S_\sigma|$. Then S_σ, central in $C_{\hat{G}}(\hat{y})$, must contain all p-elements of $C_{\hat{G}}(\hat{y})$. So $\hat{x}$ is central in $C_{\hat{G}}(\hat{y})$, and hence $C_{\hat{G}}(\hat{x}) = C_{\hat{G}}(\hat{y})$.

3. Proofs of the corollaries.

PROOF OF COROLLARY 1. If P is a T.I. set then $C_U(P) = C_U(\tilde{P})$ by Proposition 2.1. Assume conversely that $C_U(P) = C_U(\tilde{P})$. If S, x denotes a subset, resp. element, of G, then let $\overline{S}$, resp. $\overline{x}$, denote its image in G/U.

Now either $\overline{G}$ is a Frobenius group with kernel $\overline{P}$, or every proper normal subgroup of $\overline{G}$ contains $\overline{P}$ (see, for example, [**1**, Lemma 5.1]). In the first case, $\overline{P}$ is trivially a T.I. set in $\overline{G}$. In the second case, let J be a minimal normal subgroup of $\overline{G}$. Since J, a direct product of isomorphic simple groups, must have a cyclic Sylow p-subgroup, it follows that J is a simple group containing $\overline{P}$. Then $\overline{P}$ is a T.I. set in J, by Theorem 1. So in any event, $\overline{P}$ is a T.I. set in $\overline{G}$. Then $C_{\overline{G}}(\overline{P}) = C_{\overline{G}}(\overline{\tilde{P}})$ by Proposition 2.1.

Let $g \in C_G(\tilde{P})$. Then $\overline{g} \in C_{\overline{G}}(\overline{P})$. So if $P = \langle x \rangle$, we have $g^{-1}xg = xu$, some $u \in U$. Now xu is a p-element of UP, hence equals $v^{-1}x^iv$ for some $v \in U$ and integer i. But $xu \in xU$ and $v^{-1}x^iv \in x^iU$ imply that $x^i = x$. So $g^{-1}xg = v^{-1}xv$, some $v \in U$. Thus $vg^{-1} \in C_G(P) \leq C_G(\tilde{P})$, whence $v \in C_U(\tilde{P}) = C_U(P)$. It follows that $g \in C_G(P)$. Hence, P is a T.I. set in G by Proposition 2.1.

PROOF OF COROLLARY 2. A defect group D of the p-block containing χ may be written as $P^g \cap P$ for some $g \in G$ [**7**, III.8.14]. Then $D = \langle 1 \rangle$ or P by Theorem 1. The result follows by [**7**, IV.4.5 and VII.2.16].

PROOF OF COROLLARY 3. If $|P| = p$, the result is immediate from [**6**, Theorem 1] (see also the remarks of [**6**, p. 378] regarding the case $p < 7$). If $|P| > p$, the result follows from Corollary 2 and [**2**, Theorem 1].

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