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HERMITIAN FORMS AND THE FIBRATION OF SPHERES

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ABSTRACT. We identify the real $(2n-1)$-dimensional sphere S^{2n-1} with the unit sphere of $\mathbf{F}^2$, where $\mathbf{F}$ = reals, complexes or quaternions and $n = 1, 2$ or 4, respectively. It is shown how any Hermitian form over $\mathbf{F}^2$, restricted to S^{2n-1}, is related to the (double covering for $n = 1$, Hopf for $n = 2, 4$) fibration

$$(x_1, x_2) \to (|x_1|^2 - |x_2|^2, 2x_1\overline{x}_2): S^{2n-1} \to S^n.$$

Convexity of the joint range of several Hermitian forms over the unit sphere of an arbitrary normed vector space V over $\mathbf{F}$, with $\dim V > 2$, is deduced as a corollary.

1. Introduction. We shall present a connection between an easily constructed basis for the Hermitian forms over $\mathbf{F}^2$ and the "components" of a standard fibration of the $(2n-1)$-dimensional sphere. Here $\mathbf{F}$ is either the real field $\mathbf{R}$ $(n = 1)$, the complex field $\mathbf{C}$ $(n = 2)$ or the quaternion skew field $\mathbf{R}$ $(n = 4)$.

By a Hermitian form ϕ over a normed (left) vector space over $\mathbf{F}$ we mean a sesquilinear form evaluated on the diagonal. Thus $\phi(x) = f(x,x)$, where $f: V \times V \to \mathbf{F}$, $f(x,y)$ is linear in x and $\overline{f(x,y)} = f(y,x)$. An easy computation shows that, in the case $V = \mathbf{F}^2$, $x = (x_1, x_2)$, we may take

$$\phi(x) = [x_1 x_2] \begin{bmatrix} a & \overline{c} \\ c & b \end{bmatrix} \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \end{bmatrix} = a|x_1|^2 + b|x_2|^2 + 2\operatorname{Re}(x_1 \overline{c}\overline{x}_2) \tag{1}$$

where $(a,b,c) \in \mathbf{R}^2 \times \mathbf{F}$. (Details of these constructions are given for $\mathbf{F} = \mathbf{H}$ in §2.) There is an obvious inner product space isomorphism between $\mathbf{F}$ and n-dimensional real Euclidean space $\mathbf{E}^n$, so the forms (1) generate an $\mathbf{E}^{n+2}$.

In homogeneous form, the fibration in question has the formula

$$\Psi: x \to (|x_1|^2 + |x_2|^2, |x_1|^2 - |x_2|^2, 2x_1\overline{x}_2): \mathbf{F}^2 \to \mathbf{R}^2 \times \mathbf{F}. \tag{2}$$

Again identifying $\mathbf{R}^2 \times \mathbf{F}$ with $\mathbf{E}^{n+2}$, we shall show in §2 that the components $(\psi_1, \ldots, \psi_{n+2})$ of Ψ form an orthogonal basis for the space of Hermitian forms. Restricted to the unit sphere $\psi_1^{-1}(1)$ of $\mathbf{F}^2$, the remaining components $\hat{\Psi} = (\psi_2, \ldots, \psi_{n+2})$ decompose S^{2n-1} into fibres S^{n-1} over a base space S^n. In particular, $\hat{\Psi}$ is not homotopic to a constant map. In §3 we use this to show that the "joint range" $\Phi(U)$ is convex when $\Phi = (\phi_1, \ldots, \phi_m)$ is an m-tuple of Hermitian forms, $m \le n+1$, U is the unit sphere of V and $\dim V > 2$. §2 covers the case $\dim V = 2$, at least up to affine equivalence, as we shall see.

Some applications of, and related work on, these matters form the subject of a forthcoming survey by the author, but the following may be noted here. The convexity result is due to Brickman [**4**] when $\mathbf{F} = \mathbf{R}$ and implies the Toeplitz-Hausdorff theorem (see [**7**]) when $\mathbf{F} = \mathbf{C}$. Our methods are related to those of

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McIntosh [7] for $\mathbf{F} = \mathbf{R}$ and Davis [5] and Atkinson [1] for $\mathbf{F} = \mathbf{C}$, but the connection between (1) and (2) is more explicit here. Au-Yeung and Poon [2] have recently shown for $\mathbf{F} = \mathbf{C}$ (and stated for $\mathbf{F} = \mathbf{H}$) that the convexity result is equivalent to earlier theorems of Bohnenblust [3] and Friedland and Loewy [6], but this route involves completely different arguments and is somewhat indirect.

ACKNOWLEDGEMENT. I am most grateful to Professor Atkinson for a copy of his notes [1] on the convexity of $\Phi(V\backslash\{0\})$ for $\mathbf{F} = \mathbf{C}$, $m = 3$ which were the starting point for my investigation.

2. Forms and fibrations. The simplest case is $\mathbf{F} = \mathbf{R}$. Then the choices

$$(a, b, c) = (1, 1, 0), (1, -1, 0) \text{ and } (0, 0, 1) \tag{3}$$

form an orthogonal basis of the real Euclidean space $\mathbf{E}^3$. In terms of (1), we have a basis for the Hermitian (i.e. symmetric) forms on $\mathbf{R}^2$. The second and third forms generate

$$\hat{\Psi}: x \to (x_1^2 - x_2^2, 2x_1x_2): \mathbf{R}^2 \to \mathbf{R}^2$$

as in (2). Restricted to $\psi_1^{-1}(1) = \{x: x_1^2 + x_2^2 = 1\}$, $\hat{\Psi}$ provides a double covering of S^1. This is a consequence of the general machinery below [10, §20.4] but can also be seen from the complex version. Indeed if we write ${}^{\mathbf{C}}y = y_1 + iy_2$ for $y \in \mathbf{R}^2$ then ${}^{\mathbf{C}}\hat{\Psi}(x) = ({}^{\mathbf{C}}x)^2$.

The next case is $\mathbf{F} = \mathbf{C}$, which is inner product isomorphic to $\mathbf{E}^2$ via

$$\langle {}^{\mathbf{C}}x, {}^{\mathbf{C}}y \rangle = x_1y_1 + x_2y_2.$$

If $z = {}^{\mathbf{C}}y$ then we write $y = {}^{\mathbf{R}}z$. Appending $(a, b, c) = (0, 0, i)$ to (3), we obtain

$$(a, b, {}^{\mathbf{R}}c) = (1, 1, 0, 0), (1, -1, 0, 0), (0, 0, 1, 0) \text{ and } (0, 0, 0, 1) \tag{4}$$

as an orthogonal basis of $\mathbf{E}^4$ representing the Hermitian forms on $\mathbf{C}^2$. From (1), the final three forms generate a mapping

$$z \to (|z_1|^2 - |z_2|^2, 2\operatorname{Re}(z_1\bar{z}_2), 2\operatorname{Im}(z_1\bar{z}_2)): \mathbf{C}^2 \to \mathbf{R}^3,$$

and the final two components are just ${}^{\mathbf{R}}(2z_1\bar{z}_2)$. Comparing this with (2), we have the real components of a mapping $\hat{\Psi}$ which, restricted to $\psi_1^{-1}(1) = \{z: |z_1|^2 + |z_2|^2 = 1\}$, is the Hopf fibration: $S^3 \to S^2$, essentially in the form given by Milnor [9, p. 102].

We need some preparation for the case $\mathbf{F} = \mathbf{H}$. A quaternion q may be represented in the form

$$q = c + jd, \qquad (c, d) \in \mathbf{C}^2,$$

where $j^2 = -1$ and $dj = j\bar{d}$. We write $\bar{q} = \bar{c} - jd$, $|q|^2 = q\bar{q}$, $2\operatorname{Re} q = q + \bar{q}$ and $\alpha(q) = c + dj$. Evidently, $\alpha(\bar{q}) = \overline{\alpha(q)}$ so

$$|\alpha(q)| = |q| \quad \text{and} \quad \alpha^2(q) = q. \tag{5}$$

We also write

$${}^{\mathbf{R}}q = ({}^{\mathbf{R}}c, {}^{\mathbf{R}}d) \in \mathbf{R}^4.$$

This makes $\mathbf{H}$ isomorphic (in an obvious inner product) to $\mathbf{E}^4$, with $|q| = \|{}^{\mathbf{R}}q\|$.

Appending $(a,b,c)=(0,0,j)$ and $(0,0,ji)$ to the choices (4) for $\mathbf{C}$, we obtain an orthogonal basis of elements $(a,b,{}^{\mathbf{R}}c)$ for $\mathbf{E}^6$ representing the Hermitian forms on $\mathbf{H}^2$. The final five forms (1) generate a mapping

$$\begin{aligned}\Omega\colon \mathbf{H}^2 \to \mathbf{R}^5\colon (q_1,q_2) &= (c_1+jd_1, c_2+jd_2)\\ &\to (|q_1|^2-|q_2|^2, 2\,\mathrm{Re}(c_1\bar{c}_2+\bar{d}_1d_2),\\ &\qquad 2\,\mathrm{Im}(c_1\bar{c}_2-\bar{d}_1d_2), 2\,\mathrm{Re}(\bar{c}_2\bar{d}_1-c_1d_2), 2\,\mathrm{Im}(c_1d_2+\bar{c}_2\bar{d}_1)).\end{aligned}$$

The final four components of $\Omega(q_1,q_2)$ equal ${}^{\mathbf{R}}p$ where

$$p = 2(c_1\bar{c}_2+d_1\bar{d}_2)+2j(\bar{c}_2\bar{d}_1-\bar{c}_1\bar{d}_2) = 2\alpha(q_1)\overline{\alpha(q_2)}.$$

Thus Ω furnishes the real components of $\hat{\Psi}\circ\alpha$, $\hat{\Psi}$ as in (2). We have almost established the following

THEOREM 1. *Let n be the dimension of $\mathbf{F}$, considered as a real vector space. Then a basis $\psi_1,\ldots,\psi_{n+2}$ for the Hermitian forms over $\mathbf{F}^2$ exists which, restricted to the unit sphere, has the following properties:* (i) *ψ_1 has constant value* 1, (ii) *$\hat{\Psi}=(\psi_2,\ldots,\psi_{n+2})$ provides a fibration of S^{2n-1} with base space S^n and fibres S^{n-1}.*

To complete the argument for $\mathbf{F}=\mathbf{H}$, we denote the one-point compactification of $\mathbf{H}$ by $\mathbf{H}\cup\{\infty\}$, and, defining $\alpha(\infty)=\infty$, we readily verify via (5) that α is a homeomorphism of $\mathbf{H}\cup\{\infty\}$ onto itself. This allows the argument of [**8**, p. 67 or **10**, p. 109] for the Hopf map to go through for $\hat{\Psi}\circ\alpha$—again we are identifying the restriction of $\hat{\Psi}$ to $\psi_1^{-1}(1)=\{q\colon |q_1|^2+|q_2|^2=1\}$ with the Hopf map $S^7\to S^4$ (cf. [**9**, p. 102]).

3. Convexity of joint ranges. As in the introduction, let V be a normed vector space over $\mathbf{F}$, with $U=\{x\in V\colon \|x\|=1\}$, and let $\Phi=(\phi_1,\ldots,\phi_m)$ be an m-tuple of Hermitian forms on V. For notational convenience we view α as the identity on $\mathbf{R}$ and $\mathbf{C}$. This section is devoted to the following result:

THEOREM 2. (i) *If* $\dim V=1$ *then $\Phi(U)$ is a point.*

(ii) *If* $\dim V=2$ *then $\Phi(U)$ is an affine image of S^n and is thus either a* (*convex*) *affine disc or a* (*nonconvex*) *affine sphere. In the latter case, $m\geq n+1$.*

(iii) *If* $\dim V>2$ (*e.g.* $\dim V$ *is infinite*) *and $m\leq n+1$ then $\Phi(U)$ is convex.*

PROOF. (i) follows from the representation $\phi_j(x)=a_j|x|^2$—cf. (1) with $x_2=0$.

(ii) follows mostly from Theorem 1. Evidently $\phi_j|_U$ is an affine combination of the real components of $\hat{\Psi}\circ\alpha$ which maps onto S^n. The final contention follows from the fact that $m<n+1$ forces the affine map $\Phi(U)\to S^n$ to be singular.

(iii) By appending zero forms if necessary, we may assume without loss of generality that $m=n+1$.

Suppose $\Phi(U)$ is not convex. Then there are $u_j\in U$ and a point b on the line segment joining $\Phi(u_1)$ to $\Phi(u_2)$ such that

$$b\notin\Phi(U). \tag{6}$$

Let T be the span of u_1 and u_2. By (6), $\Phi(U\cap T)$ is nonconvex, so by (i) and (ii), $\dim T=2$. By Theorem 1, an affine isomorphism exists under which $\Phi|_T$ becomes $\hat{\Psi}\circ\alpha$. For notational simplicity we shall therefore assume $\Phi|_T=\hat{\Psi}\circ\alpha$.

Let $u_3 \in U\backslash T$ and let W be the set of elements $w = \sum_{j=1}^{3} a_j u_j \in U$ such that a_3 is nonnegative real. Since $T \cap U$ is an S^{2n-1}, W is homeomorphic to a hemisphere bounded by S^{2n-1}, i.e. to a real $2n$-dimensional ball B^{2n}. Thus $\Phi|_W$ is a continuous extension of $\hat{\Psi} \circ \alpha$ from S^{2n-1} to B^{2n}. In particular, $\hat{\Psi} \circ \alpha$ is homotopic to a constant, and this is a contradiction [**8**, p. 67, Proposition 5.1].

REFERENCES

1. F. V. Atkinson, *On the range of a set of three Hermitian quadratic forms*, unpublished, 1981.
2. Y.-H. Au-Yeung and Y.-T. Poon, *A remark on the convexity and positive definiteness concerning Hermitian matrices*, Southeast Asian Bull. Math. **3** (1979), 85–92.
3. F. Bohnenblust, *Joint positiveness of matrices*, unpublished.
4. L. Brickman, *On the field of values of a matrix*, Proc. Amer. Math. Soc. **12** (1961), 61–66.
5. C. Davis, *The shell of a Hilbert-space operator*. II, Acta Sci. Math. (Szeged) **31** (1970), 301–318.
6. S. Friedman and R. Loewy, *Subspaces of symmetric matrices containing matrices with a multiple first eigenvalue*, Pacific J. Math. **62** (1976), 389–399.
7. A. McIntosh, *The Toeplitz-Hausdorff theorem and ellipticity conditions*, Amer. Math. Monthly **85** (1978), 475–477.
8. S.-T. Hu, *Homotopy theory*, Academic Press, New York, 1959.
9. J. Milnor, *Singular points of complex hypersurfaces*, Princeton Univ. Press, Princeton, N. J., 1968.
10. N. Steenrood, *The topology of fibre bundles*, Princeton Univ. Press, Princeton, N. J., 1951.

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