PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 94, Number 4, August 1985

A PRIME IDEAL IN A POLYNOMIAL RING WHOSE SYMBOLIC BLOW-UP IS NOT NOETHERIAN

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ABSTRACT. Let R be the polynomial ring $k[X, Y, Z]$ localized at the maximal ideal $M = (X, Y, Z)$. We construct a prime ideal P in R which is equal to the ideal of m generic lines through the origin modulo M^m, and we show that, for suitable choice of m, the symbolic blow-up of such an ideal P is not Noetherian.

If P is a prime ideal in a commutative Noetherian ring A, its nth symbolic power is the ideal

$$P^{(n)} = \{x \in A \mid sx \in P^n \text{ for some } s \notin P\}.$$

The symbolic blow-up of P is then defined to be the graded ring

$$A \oplus P \oplus P^{(2)} \oplus \cdots = \bigoplus_{n \geqslant 0} P^{(n)}.$$

In a few nice cases the symbolic blow-up of P is a Noetherian ring or, equivalently, a finitely generated A-algebra. In general, however, $\oplus P^{(n)}$ is not Noetherian. The first example of this is due to Rees [**5**]; in his example, P is a prime ideal in the graded ring of a projective curve of genus $\geqslant 1$. Another example related to this question was constructed by Nagata in his counterexample to Hilbert's fourteenth problem (see Nagata [**4**] or Dixmier [**1**]). In Nagata's example the ideal involved is not prime, but the ring A is a polynomial ring in three variables, and a similar construction (which we explain below) yields a graded ring which is not finitely generated over A. More recently, Cowsik asked whether the symbolic blow-up of a prime ideal in a regular local ring is finitely generated (see Huneke [**3**]). In this paper we show that Nagata's example can be used to construct a symbolic blow-up which is not Noetherian in which P is a prime ideal in the localization at the origin of a polynomial ring in three variables.

We begin by recalling Nagata's example. Let k be a suitably large field of characteristic zero (for example, k can be the complex numbers). Let I be the ideal in $k[X, Y, Z]$ corresponding to m generic lines through the origin. More precisely, take $3m$ elements of k which are independent over the ground field and denote them by a_i, b_i, c_i for $i = 1, \ldots, m$. Then let $\alpha_i = a_iY - b_iX$ and $\beta_i = a_iZ - c_iX$. Let I_i be the ideal generated by α_i and β_i, and let $I = \bigcap_{i=1}^m I_i$. We now localize at the origin; let

Received by the editors, June 4, 1984.
1980 *Mathematics Subject Classification*. Primary 13E05, 13F20.
[1]Supported in part by a Grant from the National Science Foundation.

$A = k[X, Y, Z]$ localized at the maximal ideal $M = (X, Y, Z)$ and retain the notation I_i and I for the corresponding ideals in A.

Note that the dimension of A/I is one. In this context we define the nth symbolic power as follows:

DEFINITION. If I is an ideal in a local ring A with maximal ideal M, and if the dimension of A/I is one, then the nth symbolic power, denoted $I^{(n)}$, is $\{x \in A \mid M^k x \subseteq I^n$ for some integer $k\}$.

If I is prime, this definition is equivalent to the usual one. In Nagata's example, $I^{(n)} = \bigcap_{i=1}^m I_j^n$. Using this definition, Nagata [**4**] proves that if m is a perfect square $\geqslant 16$, $\oplus I^{(m)}$ is not Noetherian.

Let $\hat{A}$ be the completion of A, and, for any module S, let $\hat{S} = S \otimes \hat{A}$. We will construct a prime ideal P in A such that $\dim A/P = 1$ and such that $\hat{P}$ is isomorphic to $\hat{I}$ as an ideal of $\hat{A}$.

PROPOSITION 1. *Let J be an ideal in a local ring A with* $\dim(A/J) = 1$. *Then $\oplus J^{(n)}$ is finitely generated over A if and only if $\oplus \hat{J}^{(n)}$ is finitely generated over $\hat{A}$.*

PROOF. Using the faithful flatness of $\hat{A}$ over A, the result will follow if we can show that $\hat{J}^{(n)} = J^{(n)} \otimes \hat{A}$ for all n. To see this, first note that $\hat{J}^n = J^n \otimes \hat{A}$. For any A-module S, if we let $\Gamma_M(S) = \{x \in S \mid M^k x = 0$ for some integer $k\}$, then

$$\Gamma_M(S \otimes \hat{A}) = \Gamma_M(S) \otimes \hat{A}$$

as submodules of $S \otimes \hat{A}$. Applying this to A/J^n, we get $J^{(n)}/J^n \otimes \hat{A} = \hat{J}^{(n)}/\hat{J}^n$ as submodules of $\hat{A}/\hat{J}^n$, so $J^{(n)} \otimes \hat{A} = \hat{J}^{(n)}$.

It follows from Proposition 1 that if P satisfies $\hat{P} \cong \hat{I}$ as ideals of $\hat{A}$, then $\oplus P^{(n)}$ will not be Noetherian. The construction of an ideal with this property is essentially geometric; we find a curve whose tangent cone at the origin is the union of the generic lines defined by the ideals I_i. As we show below, if an ideal P with this property is congruent to I modulo a sufficiently high power of M, then $\hat{P}$ will be isomorphic to $\hat{I}$. We first construct a curve in $\mathbf{P}^4$ and then project down to give the required curve in $\mathbf{A}^3$.

Let V be a smooth hypersurface of degree m in $\mathbf{P}^4$, and let L be a line which meets V in m distinct points $q_1, \ldots, q_m$. Let p and r be two more points on L not equal to any of the q_i's. Let H be a hyperplane which meets L only at the point p, and let f: $V \to H$ be the map obtained by projecting from r. Then $f^{-1}(p) = \{q_1, \ldots, q_m\}$, and, since V must meet L transversally at each point q_i, the induced maps f^*: $O_{p,H} \to O_{q_i,V}$ induce isomorphisms on the completions $\hat{O}_{p,H} \to \hat{O}_{q_i,V}$. We now identify $O_{p,H}$ with the ring A defined above. For each $i = 1, \ldots, m$ the ideal $I_i \subseteq A$ defines an ideal $(f^*(\alpha_i), f^*(\beta_i))$ in $O_{q_i,V}$. We wish to construct a curve $C \subseteq V$ defined by homogeneous polynomials g and h such that

(a) C passes through q_i for each i;

(b) in the maximal ideal M_{q_i} we have $g \equiv f^*(\alpha_i)$ and $h \equiv f^*(\beta_i) \bmod M_{q_i}^m$.

To see that this can be done, let $\mathscr{I}_q$ be the sheaf of ideals of functions in O_V vanishing at $q_i, \ldots, q_m$. We then have an exact sequence of sheaves

$$0 \to \mathscr{K} \to \mathscr{I}_q \xrightarrow{\phi} \oplus\, M_{q_i}/M_{q_i}^m \to 0,$$

where $\mathscr{K}$ is the kernel of ϕ. If we tensor this with $O_V(n)$ and use the fact that $\oplus M_{q_i}/M_{q_i}^m$ is supported at a finite number of points, we have

$$0 \to \mathscr{K}(n) \to \mathscr{I}_q(n) \to \oplus M_{q_i}/M_{q_i}^m \to 0.$$

For n large enough, $H^1(V, K(n)) = 0$, so the map on global sections is surjective, and we can choose a curve satisfying (a) and (b). We must show that g and h can be chosen so that the curve C is irreducible. In fact, by Bertini's Theorem (see Hartshorne [**2**, Corollary III.10.9]), it is enough to show that, for n large enough, the linear system of sections of $\mathscr{I}_q(n)$ whose image in $\oplus M_{q_i}/M_{q_i}^m$ is a constant multiple of $f^*(\alpha_i)$ has no base points except $q_i, \ldots, q_m$; a generic member will then define a smooth surface F, and, repeating the argument with F, we can choose g and h for which the curve is smooth and, hence (see Hartshorne [**2**, Remark III.10.9.1]), irreducible.

Choose n so that $\mathscr{I}_q(n) \to \oplus M_{q_i}/M_{q_i}^m$ is surjective and $n \geqslant m^2$. Let s be a section of $\mathscr{I}_q(n)$ with the correct image in $\oplus M_{q_i}/M_{q_i}^m$, and let q be a point in V not equal to any q_i. If $s(q) \neq 0$, q is not a base point. If $s(q) = 0$, let t_i be a linear function (i.e., a section of $O_V(1)$) which vanishes at q_i but not at q for each $i = 1, \ldots, m$. Let t be a section of $O_V(n - m^2)$ which does not vanish at q. Let $s' = s + t\Pi t_i^m$. Then s' is a section of $\mathscr{I}_q(n)$ which has the same image as s in $\oplus M_{q_i}/M_{q_i}^m$, but does not vanish at q. Thus the only base points of our linear system are $q_i, \ldots, q_m$. However, a generic element is automatically smooth at $q_i, \ldots, q_m$, so by Bertini's Theorem, a generic choice of C will be smooth and, hence, irreducible.

Let B be the semilocal ring of the points $q_1, \ldots, q_m$ in V, and let $N = M_1 \cap \cdots \cap M_m$ be the intersection of its maximal ideals. Let J be the ideal of B generated by g and h. Let $P = (f^*)^{-1}(J)$, where f^* is the map induced by $f\colon V \to H$, so that P is the prime ideal at the origin corresponding to the projection of C onto H. We claim that $\hat{P} \cong \hat{I}$ as ideals of $\hat{A}$. To show this it will suffice to show that $\hat{A}/\hat{P} \cong \hat{A}/\hat{I}$; then, since $\hat{A}$ is a power series ring, this isomorphism can be lifted to an automorphism of $\hat{A}$ taking $\hat{P}$ to $\hat{I}$.

We first show that $\hat{P}$ is the intersection of ideals $\hat{P}_i$, each generated by two elements which are independent modulo $\hat{M}^2$ and such that $\hat{P}_i + \hat{M}^m = \hat{I}_i + \hat{M}^m$. Consider the map $\hat{f}^*\colon \hat{A} \to \hat{B}$ induced by f, where $\hat{B}$ is the completion of B at N. Since B is a semilocal ring with maximal ideals $M_1, \ldots, M_m$, we have

$$\hat{B} \cong \prod_{i=1}^{m} \hat{B}_i,$$

and $\hat{f}^*$ induces an isomorphism from $\hat{A}$ to $\hat{B}_i$ for each i. Furthermore, if $\hat{J}_i$ denotes the image of the ideal J of the curve C in $\hat{B}_i$, then $\hat{J}_i$ is generated by two elements congruent to $\hat{f}^*(\alpha_i)$ and $\hat{f}^*(\beta_i)$ modulo $\hat{M}_i^m$. It follows that $\hat{J} = \bigcap \hat{J}_i$, so we have

$$\hat{P} = (\hat{f}^*)^{-1}(\hat{J}) = \bigcap (\hat{f}^*)^{-1}(\hat{J}_i) = \bigcap \hat{P}_i,$$

where $\hat{P}_i$ is generated by two elements congruent to α_i and β_i modulo $\hat{M}^m$. Thus $\hat{P} + \hat{M}^m = \hat{I} + \hat{M}^n$, and we have a commutative diagram

$$\begin{array}{ccc} \hat{A}/(\hat{P} + \hat{M}^n) & \xrightarrow{\sim} & \hat{A}/(\hat{I} + \hat{M}^n) \\ \downarrow & & \downarrow \\ \prod_{i=1}^m \hat{A}/(\hat{P}_i + \hat{M}_i^m) & \xrightarrow{\sim} & \prod_{i=1}^m \hat{A}/(\hat{I}_i + \hat{M}_i^m) \end{array}$$

where the horizontal isomorphisms are induced by identity maps. Since the rings on the bottom are products of power series rings in one variable, the isomorphisms can be lifted to an isomorphism ψ from $\prod_{i=1}^m \hat{A}/\hat{P}_i$ to $\prod_{i=1}^m \hat{A}/\hat{I}_i$. We claim that such an isomorphism will take the image of $\hat{A}/\hat{P}$ to the image of $\hat{A}/\hat{I}$; this will complete the proof.

To show this it suffices to show that the maps from $\hat{M}^m(\hat{A}/\hat{P})$ to $\prod \hat{M}_i^m(\hat{A}_i/\hat{P}_i)$ and from $\hat{M}^m(\hat{A}/\hat{I})$ to $\prod \hat{M}_i^m(\hat{A}_i/\hat{I}_i)$ are surjective; since the isomorphism ψ is the identity modulo $\hat{M}^m$, it will have to map $\hat{A}/\hat{P}$ into $\hat{A}/\hat{I}$.

PROPOSITION 2. *Let $J_1,\ldots,J_m$ be distinct ideals of $A = k[[X, Y, Z]]$ such that each J_k is generated by two elements which are independent modulo M^2. Then $A/(\bigcap_{i=1}^m J_i) \to \prod_{i=1}^m A/J_i$ induces an isomorphism from $M^k(A/\bigcap_{i=1}^m J_i)$ onto $\prod_{i=1}^m M^k(A/J_i)$ for all $k \geqslant m-1$.*

PROOF. We prove this by induction on m. If $m = 1$ the assertion is clear. For $m > 1$ consider the exact sequence

$$0 \to A/K \cap J_m \to A/K \times A/J_m \to A/(K + J_m) \to 0,$$

where $K = \bigcap_{i=1}^{m-1} J_i$. Now A/J_m is a discrete valuation ring, and, for each $i \leqslant m-1$, we can find an element $x_i \in J_i$ which generates the maximal ideal of A/J_m. Then $x = \prod_{i=1}^{m-1} x_i \in \bigcap_{i<m} J_i$, and x generates M^{m-1} modulo J_m. Thus M^{m-1} annihilates $A/(K + J_m)$, so, for $k \geqslant m-1$, the map

$$M^k(A/K \cap J) \to M^k(A/K) \times M^k(A/J_m)$$

is an isomorphism. By induction, for $k \geqslant m-2$, the map

$$M^k\left(\frac{A}{K}\right) \to \prod_{i=1}^{m-1} M^k\left(\frac{A}{J_i}\right)$$

is an isomorphism. Composing the two maps, we obtain the desired result.

Thus we have shown that there is an isomorphism of $\hat{A}$ which takes $\hat{I}$ to $\hat{P}$, so $\bigoplus P^{(n)}$ is not Noetherian.

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