PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 94, Number 4, August 1985

TEST MODULES FOR PROJECTIVITY

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ABSTRACT. Let R be a commutative noetherian local ring with identity. Modules over R will be assumed to be finitely generated and unitary. A nonzero R-module M is said to be a strong test module for projectivity if the condition $\mathrm{Ext}^1_R(P, M) = (0)$, for an arbitrary module P, implies that P is projective. This definition is due to Mark Ramras [5]. He proves that a necessary condition for M to be a strong test module is that depth $M \leqslant 1$. This is also easy to see. In this note it is proved that, over a regular local ring, this condition is also sufficient for M to qualify as a strong test module.

By R we mean a commutative noetherian local ring with identity. Module means a finitely generated unitary R-module. We use [6] as a standing reference for unexplained terms and basic facts on local algebra.

In [5], Mark Ramras introduced the notion of strong test module.

DEFINITION. Let M be a nonzero R-module. M is said to be a strong test module if $\mathrm{Ext}^1_R(P, M) = (0)$, for an arbitrary module P, implies that P is projective.

It is known that the residue field and the maximal ideal of R are examples of strong test modules. Besides, a necessary condition for M to be a strong test module is that depth $M \leqslant 1$. This is well known and easy [5]. It is the purpose of this note to show that over a regular local ring this condition is also sufficient for M to qualify as a strong test module. This is Theorem 1. The proof depends on several lemmas which are well known. As usual * denotes duals and p.d. stands for projective dimension.

LEMMA 1. *Suppose A and B are nonzero R-modules. Let p.d. $A = n$. Then* $\mathrm{Ext}^n_R(A, B) \neq (0)$.

PROOF. See [2].

LEMMA 2. *Let A and B be modules over the regular local ring R, $B \neq (0)$. If* $\mathrm{Ext}^1_R(A, B) = (0)$, *then* $\mathrm{Ext}^1_R(A, R) = (0)$ *and the natural map* $A^* \otimes B \to \mathrm{Hom}(A, B)$ *is an isomorphism.*

PROOF. See [3].

We also need a fundamental exact sequence proved by M. Auslander [1]. Before stating it, let us explain some notations. Given A an R-module, let

$$\cdots \to F_n \to F_{n-1} \to \cdots \to F_0 \to A \to 0$$

Received by the editors August 13, 1984.

1980 *Mathematics Subject Classification*. Primary 13C10, 13D05; Secondary 13H10.

Key words and phrases. Projective dimension, Ext, Tor, commutative noetherian local ring, depth, Krull dimension, artinian module.

be a free resolution of A. We let $F_{-1} = A$. Set $\Omega^0 A = A, \Omega^p A = \text{Kernel}(F_{p-1} \to F_{p-2})$ and $D\Omega^p A = \text{Cokernel}(F_p^* \to F_{p+1}^*)$. Although these definitions depend on the chosen resolution of A, still they are well defined up to "projective equivalence." A consequence will be that the functors $\text{Ext}^i_R(D\Omega^p A,\ )$ and $\text{Tor}_i^R(D\Omega^p A,\ )$ are unambiguously defined, provided $i \geqslant 1$.

PROPOSITION 1. *For every pair of modules A, B and every integer $p \geqslant 0$, there exists an exact sequence* $\text{Tor}_2^R(D\Omega^p A, B) \to \text{Ext}^p_R(A, R) \otimes B \to \text{Ext}^p_R(A, B) \to \text{Tor}_1^R(D\Omega^p A, B) \to 0$.

PROOF. See [**1**].

Finally, we quote a theorem of S. Lichtenbaum which means that modules over regular local rings are "rigid."

PROPOSITION 2. *Let R be a regular local ring and let M, N be any pair of R-modules. If* $\text{Tor}_i^R(M, N) = (0)$ *for some* $i \geqslant 1$, *then* $\text{Tor}_j^R(M, N) = (0)$ *for all* $j \geqslant i$.

PROOF. See [**4**].

An easy consequence will be

PROPOSITION 3. *Let M, N be modules over the regular local ring R. Assume $N \neq (0)$ and that* depth $N = 0$; *then* $\text{Tor}_1^R(M, N) = (0)$ *implies that M is free.*

PROOF. See [**4**].

THEOREM 1. *Let R be a regular local ring. A nonzero R-module M is a strong test module, if and only if* depth $M \leqslant 1$.

PROOF. What remains to prove is that depth $M \leqslant 1$ implies M is a strong test module. Accordingly, let P be an arbitrary module satisfying $\text{Ext}^1_R(P, M) = (0)$.

Krull dimension is abbreviated as dim. If $\dim R = 0$, R is a field and any R-module is projective. Next, if $\dim R = 1$, then p.d. $P \leqslant 1$. Lemma 1 rules out p.d. $P = 1$. Hence, P is projective. Hereafter, we are permitted to assume $\dim R \geqslant 2$. We distinguish two cases according as depth $M = 0$ or depth $M = 1$.

Case (i). depth $M = 0$. Proposition 1 implies $\text{Tor}_1^R(D\Omega^1 P, M) = (0)$. By Proposition 3, $D\Omega^1 P$ is free. Since $\Omega^1 P$ is projectively equivalent to $DD\Omega^1 P$, $\Omega^1 P$ must be free, i.e. p.d. $P \leqslant 1$. Referring to Lemma 1, we conclude that P is free.

Case (ii). depth $M = 1$. This implies that the maximal ideal of R is not associated to the zero submodule of M. Suppose $\mathfrak{Z}_1, \mathfrak{Z}_2, \ldots, \mathfrak{Z}_r$ is any finite set of prime ideals each different from m. Then by standard arguments one can find an element x in $m \setminus m^2$, which is outside of every one of the prime ideals $\mathfrak{Z}_1, \mathfrak{Z}_2, \ldots, \mathfrak{Z}_r$. Taking, for $\mathfrak{Z}_1, \mathfrak{Z}_2, \ldots, \mathfrak{Z}_r$, the associated prime ideals of the zero submodule of M, such an element x has the properties (i) x is not a zero divisor on M, (ii) R/xR is a regular local ring. For a R-module Λ, denote $\overline{\Lambda} = \Lambda/x\Lambda$. We first show that $\overline{P}$ is $\overline{R}$-projective. Lemma 2 means that $\text{Ext}^1_R(P, R) = (0)$ and that the natural map

(1) $$P^* \otimes M \to \text{Hom}_R(P, M) \cdots$$

is an isomorphism. A further consequence is that the sequence

$$0 \to P^* \xrightarrow{x} P^* \to \overline{P}^* \to 0$$

is exact, where the dual at the right extreme is over $\overline{R}$. This results by applying $\mathrm{Ext}_R(P,\ )$ to the exact sequence $0 \to R \xrightarrow{x} R \to \overline{R} \to 0$. This means, tensoring with M, the sequence

$$P^* \otimes_R M \xrightarrow{x} P^* \otimes_R M \to \overline{P}^* \otimes_R M \to 0$$

is exact. Next, applying $\mathrm{Ext}_R(P,\ )$ to the exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$ and making use of $\mathrm{Ext}^1_R(P, M) = (0)$, we find

$$0 \to \mathrm{Hom}_R\,(P, M) \xrightarrow{x} \mathrm{Hom}_R\,(P, M) \to \mathrm{Hom}_{\overline{R}}(\overline{P}, \overline{M}) \to 0$$

is exact. Noticing that $\overline{P}^* \otimes_R M \cong \overline{P}^* \otimes_{\overline{R}} \overline{M}$, we get the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & P^* \otimes_R M & \to & P^* \otimes_R M & \to & \overline{P}^* \otimes_{\overline{R}} \overline{M} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Hom}_R(P, M) & \to & \mathrm{Hom}_R(P, M) & \to & \mathrm{Hom}_{\overline{R}}(\overline{P}, \overline{M}) & \to & 0.
\end{array}$$

The first two vertical maps from the left are isomorphisms in view of (1). Hence by the snake lemma, the natural homomorphism

$$\overline{P}^* \otimes_{\overline{R}} \overline{M} \to \mathrm{Hom}_{\overline{R}}(\overline{P}, \overline{M})$$

is an isomorphism of $\overline{R}$-modules. Referring to Proposition 1, we conclude that $\mathrm{Tor}_1^{\overline{R}}(D\overline{P}, \overline{M}) = (0)$. Since $\overline{R}$ is regular local and depth $\overline{M} = 0$, $D\overline{P}$ must be $\overline{R}$-projective by Proposition 3. From this it follows easily that $\overline{P}$ is $\overline{R}$-projective.

To complete the proof, we can certainly assume $P \neq (0)$. Now P cannot be an artinian module; for if it were, then so would $\overline{P}$ be, as an $\overline{R}$-module; being free over $\overline{R}$ and $\dim \overline{R} \geqslant 1$, we would arrive at $\overline{P} = (0)$, i.e. $P = xP$ and so $P = (0)$, by Nakayama lemma, which would be a contradiction. Let L be the maximal artinian submodule of P and let N be the quotient P/L. Note that $N \neq (0)$ and depth $N > 0$. Using $\mathrm{Ext}^1_R(P, M) = (0)$ in the exact sequence $0 \to L \to P \to N \to 0$, we get a surjective homomorphism

$$\mathrm{Hom}(L, M) \to \mathrm{Ext}^1_R(N, M) \to 0 \cdots . \tag{2}$$

The data, depth $M = 1$, L is artinian immediately yields $\mathrm{Hom}(L, M) = (0)$. Hence, from (2), we find that $\mathrm{Ext}^1_R(N, M) = (0)$. Now since depth N and depth M are both positive, by the procedure employed in the beginning of the proof of case (ii), we can find $y \in m \setminus m^2$ which is not a zero divisor of both N and M. Then starting from $\mathrm{Ext}^1_R(N, M) = (0)$, we find as before, that N/yN is a projective R/yR-module. Now the additional information that y is not a zero divisor on N clearly means that N is R-projective. Hence, the sequence, $0 \to L \to P \to N \to 0$ splits and so there is an isomorphism $P = L \oplus N$, i.e., $\overline{P} = \overline{L} \oplus \overline{N}$ (notice that $\overline{P} = P/xP$, etc.), $\overline{P}$ being $\overline{R}$-free, $\overline{L}$ is $\overline{R}$-free as well. $\overline{L}$ being a finite length $\overline{R}$-module, this is possible only when $\overline{L} = (0)$, i.e. only when $L = (0)$, by Nakayama lemma. Hence $P = N$, i.e., P is free R-module. This finishes the proof.

COROLLARY. *Let R be a regular local ring. Let M be a nonzero R-module. For an arbitrary R-module P, and integer $i \geqslant 1$, the condition* $\operatorname{Ext}^i_R(P, M) = (0)$ *implies p. d. $P < i$ if and only if* depth $M \leqslant 1$.

PROOF. This follows from Theorem 1 by easy dimension shifting argument.

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