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# A CHARACTERIZATION OF THE INVARIANT SUBSPACES OF DIRECT SUMS OF STRICTLY CYCLIC ALGEBRAS

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ABSTRACT. Two characterizations of the invariant subspace lattice of $A^{(n)}$ for a strictly cyclic operator algebra $A$ on a separable Hilbert space are proven.

We derive a characterization of the invariant subspaces of $A^{(n)}$, where $A$ is an Abelian, hereditarily strictly cyclic operator algebra on a separable Hilbert space. A further characterization is derived in the case when $A$ is generated by a single operator. The first result is also generalized to $A^{(\infty)}$ when $A$ has Donoghue lattice.

We will need some notation and definitions. Throughout this paper, $H$ will be a separable, infinite-dimensional, complex Hilbert space, and $B(H)$ the algebra of all bounded linear operators on $H$. If $T \in B(H)$, then $A(T)$ is the weakly closed subalgebra of $B(H)$ generated by $T$ and the identity. If $F \subset B(H)$, then Lat $F$ is the lattice of all invariant subspaces of $F$. We call $F$ *transitive* if Lat $F = \{\{0\}, H\}$, and we call $F$ *unicellular* if Lat $F$ is totally ordered. By $H^{(n)}$ we mean the direct sum of $n$ copies of $H$. For $T \in B(H)$, $T^{(n)}$ is the operator on $H^{(n)}$ defined by

$$T^{(n)}(x_1, x_2, \ldots, x_n) = (Tx_1, Tx_2, \ldots, Tx_n),$$

and $F^{(n)} = \{T^{(n)}\colon T \in F\}$. If $M$ is a subspace of $H^{(n)}$, the $i$th *kernel* of $M$ is the collection of all vectors in $M$ whose $i$th coordinate is 0. Note that if $M \in \operatorname{Lat} T^{(n)}$, then the $i$th kernel of $M$ is invariant under $T^{(n)}$, and it is obviously isomorphic to an element of $\operatorname{Lat} T^{(n-1)}$. If $F^{(n)} \subset B(H^{(n)})$, and if $M \in \operatorname{Lat} F^{(n)}$, then $M$ is an *invariant graph subspace of $F^{(n)}$ on the ith coordinate* if $M$ has the form

$$M = \{(T_1x, T_2x, \ldots, T_{\lambda-1}x, x, T_{i+1}x, \ldots, T_nx)\colon x \in D\}$$

for some linear manifold $D$ of $H$ and for linear transformations $T_i$ with domain $D$ and range contained in $H$. The $T_i$'s will be called *graph transformations for $F$*. In general, the $T_i$'s need not be closed, although jointly they are closed since $M$ is closed. If $M$ and $N$ are subspaces of $H$, we use $M \vee N$ for the (closed linear) span of $M$ and $N$.

Note that if $M$ is an invariant subspace of $F^{(n)}$, then $M$ is a graph subspace on the $i$th coordinate if and only if the $i$th kernel is $\{0\}$; i.e., if and only if the $i$th coordinate of a vector determines the vector. Note also that the domain of a graph transformation for $F$ is invariant under $F$, and the transformation commutes with

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every operator in $F$. In particular, if $T$ is a graph transformation, so is $T - \lambda$ for any scalar $\lambda$.

Recall that a subalgebra $A$ of $B(H)$ has *finite strict multiplicity* if there is a finite collection of vectors $E = \{x_1, x_2, \ldots, x_n\}$ such that

$$\{T_1x_1 + T_2x_2 + \cdots + T_nx_n: T_i \in A\} = H.$$

The minimal cardinality of all such sets $E$ is called the *strict multiplicity of* $A$. If $A$ has strict multiplicity 1, $A$ is said to be *strictly cyclic*. An operator $T$ has *finite strict multiplicity* if $A(T)$ does, and $T$ is *strictly cyclic* if $A(T)$ is. An operator algebra $A$ is said to be *hereditarily strictly cyclic* if the uniform closure of its restriction to every invariant subspace is strictly cyclic. The vector $x$ is a *separating vector* for $A$ if $Tx = 0$ implies $T = 0$ whenever $T$ is an operator in $A$. Note that cyclic vectors for Abelian operator algebras are always separating.

The following propositions are known results which we will need.

PROPOSITION 1 (HERRERO [7]). *A uniformly closed operator algebra of finite strict multiplicity has no dense invariant manifolds other than $H$.*

PROPOSITION 2 (LAMBERT [**11**]). *If $T$ is an operator in an Abelian strictly cyclic algebra, then the spectrum of $T$ consists entirely of compression spectrum.* (*Recall that the compression spectrum of $T$ is the set of complex numbers $\lambda$ such that the range of $T - \lambda$ is not dense in $H$.*)

PROPOSITION 3 (HERRERO [**8**]). *Every densely defined linear transformation commuting with every operator in an algebra of finite strict multiplicity is bounded.*

PROPOSITION 4 (ROSENTHAL [**20**]). *Let $T$ be strictly cyclic. If $T$ is unicellular, then $T$ has one-point spectrum. Conversely, if $T$ is hereditarily strictly cyclic, and if $T$ has one-point spectrum, then $T$ has Donoghue lattice* (*i.e., there is an orthonormal basis $\{e_i\}_{i=0}^{\infty}$ of $H$ such that the nontrivial invariant subspaces of $T$ are the subspaces $M_k = \bigvee_{i=k}^{\infty} e_i$ for positive integers $k$*).

THEOREM 1. *Let $A$ be an Abelian, unicellular, hereditarily strictly cyclic algebra. Then every element of* Lat $A^{(n)}$ *can be expressed as the span of at most $n$ invariant graph subspaces whose domains are in* Lat $A$.

PROOF. Let $M \in$ Lat $A^{(n)}$, and let $P_i$ be the $i$th coordinate projection. Since Lat $A$ is totally ordered and each $\overline{P_iM}$ (the closure of $P_iM$) is in Lat $A$, we can choose $i_0$ such that $P_iM \subset \overline{P_{i_0}M}$ for $i = 1, 2, \ldots, n$.

CLAIM. $P_{i_0}M$ is closed.

PROOF OF CLAIM. Let $\tilde{A}$ be the closure of $A/\overline{P_{i_0}M}$, and let $N = \overline{P_{i_0}M}^{(n)}$. The algebra $A$ is strictly cyclic by hypothesis, and, obviously, $\tilde{A}^{(n)}$ is the (uniform) closure of $A^{(n)}/N$. In particular, $M \in$ Lat $\tilde{A}^{(n)}$, so $P_{i_0}M$ is invariant under $\tilde{A}$. Of course, $P_{i_0}M$ is dense in $\overline{P_{i_0}M}$, and so by Proposition 1, $P_{i_0}M = \overline{P_{i_0}M}$, and the claim is proven.

The above conclusion implies that we can choose a vector $x_0 \in P_{i_0}M$ which is strictly cyclic for $\tilde{A}$. Choose $f_0 \in M$ such that $P_{i_0}f_0 = x_0$, and let

$$G_0 = A^{(n)}f_0, \qquad G = \tilde{A}^{(n)}f_0.$$

Every vector in $G$ is a limit of vectors in $G_0$ since every operator in $\tilde{A}$ is a limit of operators in $A/P_{i_0}M$. Hence, $G$ is invariant under $A^{(n)}$. If we show that $G$ is a graph on the $i_0$th coordinate, we will know that $G$ is closed since its graph transformations will be densely defined and commute with every operator in an Abelian strictly cyclic algebra; Proposition 3 then implies that they will be bounded. Since the domain of $G$ is $P_{i_0}M$, its domain is in Lat $A$.

To show that $G$ is a graph on the $i$th coordinate, it is enough to show that if $(y_1, y_2, \ldots, y_n)$ is in $G$ with $y_{i_0} = 0$, then $y_i = 0$ for $i = 1, 2, \ldots, n$. But this is clear since $x_0$ is a separating vector; i.e., if $Tx_0 = y_{i_0} = 0$, then $T = 0$, so $T^{(n)}f_0 = 0$.

Let $K$ be the $i_0$th kernel of $M$. Trivially, $K \in \text{Lat } A^{(n)}$; and if $x \in M$, there exists a $y$ in $G$ such that $P_{i_0}x = P_{i_0}y$ (since $P_{i_0}G = P_{i_0}M$), so $x - y \in K$. Thus, $y + (x - y) \in M$, so $M = G \vee K$. We can now perform the same procedure on $K$. Since $P_{i_0}K = \{0\}$, the index chosen will be different from $i_0$, and the next kernel chosen will have at least two coordinate projections which are $\{0\}$. We continue in this manner getting $n$ invariant graph subspaces of $A^{(n)}$. Since the number of zero coordinate projections increases at each step, and since $M$ is obviously the span of these subspaces, we are done.

A little more can be said about the structure of the graphs. If $G_1, G_2, \ldots, G_n$ are the graph subspaces in the order chosen in the proof, then $G_n$ is a coordinate subspace; $G_{n-1}$ is 0 on the domain slots of $G_1, \ldots, G_{n-2}$; $G_{n-2}$ is 0 on the domain slots of $G_1, \ldots, G_{n-3}$; and so on. (Of course, some of the $G_i$'s may be 0.)

Nikolskii **[15]** has shown that if $T$ is a weighted shift with $p$-summable weights $(p > 0)$ which go monotonically to 0, then $T$ has Donoghue lattice (Nikolskii actually obtains more general conditions for unicellularity). Lambert **[11]** has shown that such an operator is strictly cyclic. Since the restriction of $T$ to any invariant subspace is another such operator, the algebra generated by $T$ satisfies the hypothesis of the theorem. Nordgren **[17]** has shown that every Donoghue operator $T$ has the property that each of the invariant subspaces of $(T^*)^{(n)}$ is a span of finite-dimensional invariant subspaces. The following theorem shows that this is true for a much wider class of operators.

THEOREM 2. *Let $T$ be hereditarily strictly cyclic with $\sigma(T) = \{\lambda\}$. Then if $M \in$ Lat$(T^*)^{(n)}$, $M$ is a span of finite-dimensional invariant subspaces of $(T^*)^{(n)}$. Equivalently, if $M \in$ Lat $T^{(n)}$, $M$ is an intersection of invariant subspaces of $T^{(n)}$ of finite codimension.*

PROOF. Replacing $T$ by $T - \lambda$ if necessary, we can assume that $\lambda = 0$. By Theorem 1 we need only consider invariant graph subspaces whose domains are subspaces of $H$. So let

$$G = \{(x, T_1x, T_2x, \ldots, T_{n-1}x) : x \in D\}$$

be a graph subspace of $T^{(n)}$. The subspace $D$ is in Lat $T$ and, by Proposition 4, has the form $D = \bigvee_{i=k}^{\infty} e_i$ for an orthonormal basis $\{e\}_{i=0}^{\infty}$.

Let $M_j = \bigvee_{i=j}^{\infty} e_i$ (so $D = M_k$). Then, for each $j$, $M_j^{(n)}$ is invariant under $T^{(n)}$ and has finite codimension. Let

$$f_k = (e_k, T_1e_k, \ldots, T_{n-1}e_k) \quad \text{and} \quad f_k^i = (T^{(n)})^i f_k.$$

Then $f_k \in G$, and so $f_k^i \in G$ since $G \in \operatorname{Lat} T^{(n)}$. Since the spectrum of any strictly cyclic operator consists entirely of compression spectrum, by Proposition 2, $TM_j \subset M_{j+1}$ for every $j$. Thus, since $f_k \in M_k^{(n)}$, we have $f_k^i = (T^{(n)})^j f_k \in M_{k+j}$. Let

$$N_j = f_k \vee f_k^1 \vee f_k^2 \vee \cdots \vee f_k^j \vee M_{k+j+1}^{(n)}.$$

Then $N_j$ has finite codimension (since $M_{k+j+1}^{(n)}$ has finite codimension). Also, $N_j \in \operatorname{Lat} T^{(n)}$ since $M_{k+j+1}^{(n)} \in \operatorname{Lat} T^{(n)}$ and $T^{(n)} f_k^i = f_k^{i+1}$.

We will be done if we can show that $G = \bigcap_{j=1}^{\infty} N_j$. Now, since $\bigcap_{j=1}^{\infty} M_{k+j+1}^{(n)} = \{0\}$, it follows that

$$\bigcap_{j=1}^{\infty} N_j = f_k \vee \left( \bigvee_{i=1}^{\infty} f_k^i \right).$$

Hence, by the definition of $f_k^i$, $\bigcap_{j=1}^{\infty} N_j$ is the smallest invariant subspace of $T^{(n)}$ containing $f_k$, which is exactly the definition of $G$. This completes the proof.

In the next theorem, we extend Theorem 1. It will be convenient to introduce some terminology. We define $H^{(\infty)}$ to be the collection of all square-summable sequences of vectors from $H$, and if $T \in B(H)$, we define $T^{(\infty)}$ on $H^{(\infty)}$ by $T^{(\infty)}(x_1, x_2, \ldots) = (Tx_1, Tx_2, \ldots)$. It is straightforward to verify that $H^{(\infty)}$ is a Hilbert space, and that $T^{(\infty)}$ is bounded. If $A$ is a subalgebra of $B(H)$, then $A^{(\infty)}$ is defined to be $\{T^{(\infty)}: T \in A\}$. If $A$ has Donoghue lattice $\{\{0\}, N_n\}$, if $M \in \operatorname{Lat} A^{(\infty)}$ and $M \neq \{0\}$, and if $n$ is the largest nonnegative integer such that $N_n$ contains every coordinate projection of $M$, then $n$ will be called the *order of M*. (Such an $n$ must exist since $N_0 = H$.)

THEOREM 3. *If $A$ is an Abelian, hereditarily strictly cyclic algebra with Donoghue lattice $\{\{0\}, N_n\}$, then every element of* Lat $A^{(\infty)}$ *is a span of invariant graph subspaces.*

PROOF. Let $M \in \operatorname{Lat} A^{(\infty)}$. Choose a graph subspace $G_{0,1} \subset M$ as we did in the proof of Theorem 1. In that proof, we chose the domain of $G$ by taking a coordinate projection of $M$ which contained every other coordinate projection. Here we modify that procedure by insisting that the coordinate index $i_0$ of the domain of $G_{0,1}$ be as small as possible (there may be more than one coordinate projection of $M$ which could be chosen). As in that earlier proof, if $K_{0,1}$ is the $i_0$th kernel of $M$, $M = G_{0,1} \vee k_{0,1}$. Continuing in this manner, we get a sequence of graph subspaces and a sequence of kernels such that

$$\begin{aligned} M = G_{0,1} \vee K_{0,1} &= G_{0,1} \vee G_{0,2} \vee K_{0,2} \\ &= G_{0,1} \vee G_{0,2} \vee \cdots \vee G_{0,i} \vee K_{0,i}. \end{aligned}$$

If $\bigcap_{i=1}^{\infty} K_{0,i} = \{0\}$, we are done. If not, let $M_1 = \bigcap_{i=1}^{\infty} K_{0,i}$. Then $M = (\bigvee_{i=1}^{\infty} G_{0,i}) \vee M_1$.

CLAIM. The order of $M_1$ is greater than the order of $M$.

PROOF OF CLAIM. Let $r$ be the order of $M_1$ and $s$ the order of $M$. If $r \not> s$, i.e. if $r = s$, then there is an index $i$ such that the $i$th coordinate projection of $M_1$ is $N_s$. Since $N_s$ contains every coordinate projection of $M$, and since we chose the domain for every graph by choosing the coordinate projection with lowest possible index, the $i$th projection must have been one of the choices. But then $N_s$ could not be a projection of $M_1$. This contradiction proves the claim.

Now we proceed with $M_1$ as we did with $M$, getting a sequence of graphs and kernels so that

$$M_1 = G_{1,1} \vee K_{1,1} = G_{1,1} \vee G_{1,2} \vee \cdots \vee G_{1,i} \vee K_{1,i}.$$

If $\bigcap_{i=1}^{\infty} K_{1,i} = \{0\}$, we are done. If not, let $M_2 = \bigcap_{i=1}^{\infty} K_{1,i}$. We continue inductively, getting a sequence of graph subspaces and a sequence $\{M_i\}$ of intersections of kernels, where the order of $M_{i+1}$ is greater than the order of $M_i$ for all $i$. We have, for each $k$,

$$M = \bigvee_{i=0}^{k} \left( \bigvee_{j=1}^{\infty} G_{i,j} \right) \vee M_k.$$

Finally, since the orders of the $M_k$'s go to infinity,

$$\bigcap_{k=1}^{\infty} M_k = \{0\} \quad \text{and} \quad M = \bigvee_{i=0}^{\infty} \left( \bigvee_{j=1}^{\infty} G_{i,j} \right),$$

and the proof is complete.

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