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HIGH ORDER COEFFICIENT ESTIMATES IN THE CLASS Σ

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ABSTRACT. Estimates are given for the coefficients b_n of functions in the class Σ in terms of $\operatorname{Re}\{1 - b_1\}$. As a consequence, there is an explicit finite number λ such that $\operatorname{Re}\{\lambda b_1 - b_n\} \leqslant \lambda$.

Introduction. Let Σ be the class of functions $g(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}$ that are analytic and univalent for $|z| > 1$. The nonvanishing subclass Σ' consists of those functions $g \in \Sigma$ for which $g(z) \neq 0$, $|z| > 1$. Denote by S the familiar class of univalent analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the open unit disk. It is easy to see that f belongs to S if and only if $g(z) = 1/f(1/z)$ belongs to Σ'.

The function $k(z) = z + 1/z$ is extremal for many problems in Σ and is closely related to the Koebe function $K(z) = z(1 - z)^{-2}$ in S. Its transform

$$k\left(z^{(n+1)/2}\right)^{2/(n+1)} = z\left(1 + z^{-n-1}\right)^{2/(n+1)}$$

is extremal for the coefficient problem $\max_{\Sigma} \operatorname{Re} b_n$ for $n = 1$ and $n = 2$, but not for $n \geqslant 3$.

It was conjectured by W. E. Kirwan [3] that

$$\operatorname{Re}\{nb_1 - b_n\} \leqslant n \qquad (n \geqslant 2) \tag{1}$$

is true for all functions in the class Σ. Equality always occurs for the function k. This conjecture is known (Garabedian and Schiffer [2]) to be true for $n = 2$ and $n = 3$.

In general, we shall consider the inequality

$$\operatorname{Re}\{\lambda b_1 - b_n\} \leqslant \lambda. \tag{2}$$

Our principal result is that this inequality holds in Σ for some finite positive λ depending on n, but not on g. If $\lambda' > \lambda$, then

$$\operatorname{Re}\{\lambda' b_1 - b_n\} = \operatorname{Re}\{\lambda b_1 - b_n\} + (\lambda' - \lambda)\operatorname{Re} b_1 \leqslant \lambda + (\lambda' - \lambda) = \lambda';$$

that is, if (2) holds for some λ, then it holds for all $\lambda' > \lambda$ and equality with λ' occurs only for translations of the function k. The problem then becomes one of finding the least value λ for which (2) is true. It follows from the work in [3] that $\lambda = n$ is the least value for which inequality (2) holds when $n = 2$ or $n = 3$. We refer the reader to the articles [6,7] for further facts related to conjecture (1) and problem (2).

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The inequality (2) is contained in the following theorem.

THEOREM. *If* $g(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}$ *belongs to* Σ *and* $n \geqslant 2$, *then there exists a finite positive number* λ *such that*

$$|\operatorname{Re} b_n| \leqslant \lambda \operatorname{Re}\{1 - b_1\}. \tag{3}$$

In particular, we have $\operatorname{Re}\{\lambda b_1 - b_n\} \leqslant \lambda$.

The number λ is given explicitly in formula (7). Afterwards, there is a section containing the numbers λ for $n = 2, 3, 4, 5$ and also simple estimates for λ that depend on n.

Note that (3) may be interpreted as coefficient estimates in a neighborhood of the function k. A generalization is stated as a concluding remark.

Proof of the theorem. Our proof is similar, but not identical, to that of R. N. Pederson [**5**] for a corresponding result in the class S. In addition, we shall keep track of the constants involved.

To estimate the coefficients of $g \in \Sigma$, we can assume without loss of generality that g is a slit mapping since such mappings are dense in Σ. If w_0 is any value omitted by g, then $g - w_0$ belongs to Σ' and we may associate with its reciprocal a function $f \in S$, as in the introduction. The function f may be embedded (cf. [**1**, Chapter 3]) into a Loewner chain $f(z, t)$ which satisfies a differential equation of the form

$$\partial f/\partial t = z\left[(1 + \kappa z)/(1 - \kappa z)\right]\partial f/\partial z, \qquad t > 0,$$

where κ is a piecewise continuous function of t, $|\kappa| = 1$, and $f(z, 0) = f(z)$. In terms of $g(z, t) = 1/f(z, t)$ the differential equation has the similar form

$$\partial g/\partial t = z\left[(1 + \bar{\kappa} z)/(1 - \bar{\kappa} z)\right]\partial g/\partial z, \qquad t > 0, \tag{4}$$

where $g(z, 0) = g(z) - w_0$.

From (4) the coefficients of $g(z, t) = e^{-t}[z + \sum_{n=0}^{\infty} b_n(t) z^{-n}]$ satisfy the differential equation

$$b_n'(t) = (n + 1)b_n(t) + 2(n - 1)\kappa(t)b_{n-1}(t) + \cdots + 2\kappa^{n-1}(t)b_1(t) - 2\kappa^{n+1}(t),$$

at least for $n \geqslant 1$. Repeated integration gives an iterated integral representation for the coefficients $b_n = b_n(0)$, $n \geqslant 1$, of $g - w_0$, hence of g, in the following form:

$$b_n = \sum(-1)^{m+1} 2^m \Gamma_{j_1 j_2 \cdots j_m} \int_0^{\infty} \int_{t_1}^{\infty} \cdots \int_{t_{m-1}}^{\infty} \kappa_1^{j_1} \kappa_2^{j_2} \cdots \kappa_m^{j_m}\, dt_m \cdots dt_2\, dt_1, \tag{5}$$

where $\kappa_j = \kappa(t_j)e^{-t_j}$, $\Gamma_{j_1} = 1$, $\Gamma_{j_1 j_2 \cdots j_m} = (j_2 + \cdots + j_m - 1)(j_3 + \cdots + j_m - 1) \cdots (j_m - 1)$, and the summation is over all positive integers $j_1, j_2, \ldots, j_m$ with $j_m \geqslant 2$ and $j_1 + j_2 + \cdots + j_m = n + 1$. Of course, b_1 is simply $2\int_0^{\infty} \kappa(t)^2 e^{-2t}\, dt$. The coefficient $b_1 = 1$ if and only if $\kappa^2 \equiv 1$.

LEMMA 1. *For* $j = 1, 2, \ldots, N$, *let* θ_j *be real and let* δ_j *be an integer multiple of* π. *Then we have the sharp inequality*

$$\begin{aligned} &|\cos(\theta_1 + \theta_2 + \cdots + \theta_N) - \cos(\delta_1 + \delta_2 + \cdots + \delta_N)| \\ &\qquad \leqslant N\left[\sum_{j=1}^{N} |\cos\theta_j - \cos\delta_j|\right]. \end{aligned} \tag{6}$$

PROOF. If $N = 1$, the inequality is trivial; assume therefore that $N \geqslant 2$. Let

$$\varphi(\theta_1,\ldots,\theta_N) = N^2 - N(\cos\theta_1 + \cdots + \cos\theta_N) + \cos(\theta_1 + \cdots + \theta_N) - 1.$$

We shall show that $\varphi \geqslant 0$. If some θ_j satisfies $\cos\theta_j \leqslant 0$, then

$$\varphi \geqslant N^2 - N(N-1) - 1 - 1 = N - 2 \geqslant 0.$$

Thus we may restrict the domain of φ to the cube $|\theta_j| < \pi/2$, $1 \leqslant j \leqslant N$. The critical points of φ are found by setting

$$\partial\varphi/\partial\theta_j = N\sin\theta_j - \sin(\theta_1 + \cdots + \theta_N) = 0 \quad \text{for } j = 1,2,\ldots,N.$$

That is, $\sin\theta_j$ has the value $(1/N)\sin(\theta_1 + \cdots + \theta_N)$ for all j. Since the interval for θ_j is restricted to $(-\pi/2, \pi/2)$, we have $\theta_1 = \cdots = \theta_N$. Thus at a critical point

$$\varphi = N^2 - N^2\cos\theta_1 + \cos(N\theta_1) - 1.$$

This is nonnegative because of the inequality $1 - \cos(N\theta) \leqslant N(1 - \cos\theta)$. We conclude that $\varphi(\theta_1,\ldots,\theta_N) \geqslant 0$ at all points. This proves (6) in case all $\delta_j = 0$.

The general case of (6) follows from this special case by replacing each θ_j by $\theta_j + \delta_j$ since $|\cos(\theta + \delta) - 1| = |\cos\theta - \cos\delta|$ whenever δ is an integer multiple of π.

Both sides of (6) vanish when all θ_j and δ_j are zero. To see that the constant N on the right side cannot be improved, choose all δ_j to be zero, set all θ_j equal to ε, and send ε to zero. □

LEMMA 2. *For each real-valued function θ, there exists a function δ whose values are either* 0 *or* π *such that*

$$2|\cos\theta - \cos\delta| \leqslant [1 - \cos(2\theta)].$$

PROOF. Choose δ so that $\cos\delta = (\cos\theta)/|\cos\theta|$ if $\cos\theta \neq 0$ and $\cos\delta = 1$ if $\cos\theta = 0$. □

We represent $\kappa(t) = e^{i\theta(t)}$ in the differential equation (4) and choose $\delta(t)$ from now on as in Lemma 2. Then

$$4\int_0^\infty |\cos\theta(t) - \cos\delta(t)|e^{-2t}\,dt \leqslant 2\int_0^\infty [1 - \cos 2\theta(t)]\,e^{-2t}\,dt = \operatorname{Re}\{1 - b_1\}.$$

We will also use the identity

$$2\int_0^\infty |\cos 2\theta(t) - \cos 2\delta(t)|e^{-2t}\,dt = 2\int_0^\infty [1 - \cos 2\theta(t)]\,e^{-2t}\,dt = \operatorname{Re}\{1 - b_1\}.$$

The choice $\theta = \delta$ in (5) gives coefficients $b_n = 0$ for $n \geqslant 2$. In general, therefore, we have

$$\operatorname{Re} b_n = \sum(-1)^{m+1}2^m\Gamma_{j_1j_2\cdots j_m}$$
$$\times\int_0^\infty\int_{t_1}^\infty\cdots\int_{t_{m-1}}^\infty\left[\cos\left(\sum_{k=1}^m j_k\theta(t_k)\right) - \cos\left(\sum_{k=1}^m j_k\delta(t_k)\right)\right]$$
$$\times e^{-\sum_{k=1}^m j_k t_k}\,dt_m\cdots dt_2\,dt_1$$

for $n \geqslant 2$. Let I denote just the iterated integral. Then Lemma 1 implies

$$|I| \leqslant m \sum_{k=1}^{m} \int_0^\infty \int_{t_1}^\infty \cdots \int_{t_{m-1}}^\infty |\cos(j_k\theta(t_k)) - \cos(j_k\delta(t_k))|\, e^{-\sum_{i=1}^m j_i t_i}\, dt_m \cdots dt_2\, dt_1$$

$$= m \sum_{k=1}^{m} I_k,$$

where

$$I_k = \int_0^\infty \int_{t_1}^\infty \cdots \int_{t_{k-1}}^\infty |\cos(j_k\theta(t_k)) - \cos(j_k\delta(t_k))| \int_{t_k}^\infty \cdots \int_{t_{m-1}}^\infty \times e^{-\sum_{i=1}^m j_i t_i}\, dt_m \cdots dt_1$$

$$= \int_0^\infty \int_{t_1}^\infty \cdots e^{-\sum_{i=1}^{k-1} j_i t_i} \int_{t_{k-1}}^\infty |\cos(j_k\theta(t_k)) - \cos(j_k\delta(t_k))| \times e^{-(j_k + \cdots + j_m)t_k}\, dt_k \cdots dt_1 / B_{j_1 j_2 \cdots j_m}(k)$$

and

$$B_{j_1 j_2 \cdots j_m}(m) = 1, \qquad B_{j_1 j_2 \cdots j_m}(k) = (j_{k+1} + \cdots + j_m)(j_{k+2} + \cdots + j_m) \cdots j_m.$$

If j_k is even, then from $t_k \geqslant t_{k-1}$ and Lemma 1 we have

$$\begin{aligned}
&\int_{t_{k-1}}^\infty |\cos(j_k\theta(t_k)) - \cos(j_k\delta(t_k))| e^{-(j_k + \cdots + j_m)t_k} dt_k \\
&\quad \leqslant e^{-(j_k + \cdots + j_m - 2)t_{k-1}} \int_0^\infty |\cos(j_k\theta(t_k)) - \cos(j_k\delta(t_k))| e^{-2t_k} dt_k \\
&\quad \leqslant e^{-(j_k + \cdots + j_m - 2)t_{k-1}} (j_k/2)^2 \int_0^\infty |\cos(2\theta(t_k)) - \cos(2\delta(t_k))| e^{-2t_k} dt_k \\
&\quad = e^{-(j_k + \cdots + j_m - 2)t_{k-1}} (j_k^2/8) \operatorname{Re}\{1 - b_1\}.
\end{aligned}$$

Similarly, if j_k is odd, then

$$\begin{aligned}
&\int_{t_{k-1}}^\infty |\cos(j_k\theta(t_k)) - \cos(j_k\delta(t_k))| e^{-(j_k + \cdots + j_m)t_k} dt_k \\
&\quad \leqslant e^{-(j_k + \cdots + j_m - 2)t_{k-1}} \int_0^\infty |\cos(j_k\theta(t_k)) - \cos(j_k\delta(t_k))| e^{-2t_k} dt_k \\
&\quad \leqslant e^{-(j_k + \cdots + j_m - 2)t_{k-1}} ((j_k + 1)/2) \\
&\qquad \times \Big\{ ((j_k - 1)/2) \int_0^\infty |\cos(2\theta(t_k)) - \cos(2\delta(t_k))| e^{-2t_k} dt_k \\
&\qquad\qquad + \int_0^\infty |\cos\theta(t_k) - \cos\delta(t_k)| e^{-2t_k} dt_k \Big\} \\
&\quad \leqslant e^{-(j_k + \cdots + j_m - 2)t_{k-1}} ((j_k + 1)/2) \\
&\qquad \times \{ ((j_k - 1)/4) \operatorname{Re}\{1 - b_1\} + (1/4) \operatorname{Re}\{1 - b_1\} \} \\
&\quad = e^{-(j_k + \cdots + j_m - 2)t_{k-1}} (j_k(j_k + 1)/8) \operatorname{Re}\{1 - b_1\}.
\end{aligned}$$

If $[x]$ denotes the greatest integer less than or equal to x, then we can combine these two cases by writing

$$\int_{t_{k-1}}^{\infty} |\cos(j_k\theta(t_k)) - \cos(j_k\delta(t_k))| e^{-(j_k+\cdots+j_m)t_k}\,dt_k$$
$$\leqslant e^{-(j_k+\cdots+j_m-2)t_{k-1}}(j_k/4)[(j_k+1)/2]\,\mathrm{Re}\{1-b_1\}.$$

It follows that

$$I_k \leqslant \mathrm{Re}\{1-b_1\}(j_k/4)[(j_k+1)/2]\int_0^{\infty}\cdots\int_{t_{k-2}}^{\infty} e^{-\sum_{i=1}^{k-1} j_i t_i}$$
$$\times e^{-(j_k+\cdots+j_m-2)t_{k-1}}\,dt_{k-1}\cdots dt_1/B_{j_1j_2\cdots j_m}(k)$$
$$= \mathrm{Re}\{1-b_1\}(j_k/4)[(j_k+1)/2]/C_{j_1j_2\cdots j_m}(k),$$

where

$$C_{j_1j_2\cdots j_m}(k) = A_{j_1j_2\cdots j_m}(k)B_{j_1j_2\cdots j_m}(k)$$

and

$$A_{j_1j_2\cdots j_m}(1) = 1,$$
$$A_{j_1j_2\cdots j_m}(k) = (j_1+\cdots+j_m-2)(j_2+\cdots+j_m-2)\cdots(j_{k-1}+\cdots+j_m-2).$$

In summary, we have the estimate $|\mathrm{Re}\, b_n| \leqslant \lambda\,\mathrm{Re}\{1-b_1\}$, where

$$\lambda = \Sigma m 2^{m-2}\Gamma_{j_1j_2\cdots j_m}\sum_{k=1}^{m} j_k[(j_k+1)/2]/C_{j_1j_2\cdots j_m}(k) \tag{7}$$

and the first summation is over all positive integers $j_1, j_2, \ldots, j_m$ with $j_m \geqslant 2$ and $j_1 + j_2 + \cdots + j_m = n+1$. Therefore (3) is proved and also

$$\mathrm{Re}\{\lambda b_1 - b_n\} \leqslant \lambda\,\mathrm{Re}\, b_1 + |\mathrm{Re}\, b_n| \leqslant \lambda\,\mathrm{Re}\, b_1 + \lambda\,\mathrm{Re}\{1-b_1\} = \lambda. \quad \square$$

Estimates for λ. Direct computation of (7) leads, for example, to the estimates

$$|\mathrm{Re}\, b_2| \leqslant 8\,\mathrm{Re}\{1-b_1\}, \quad |\mathrm{Re}\, b_3| \leqslant (115/3)\,\mathrm{Re}\{1-b_1\},$$
$$|\mathrm{Re}\, b_4| \leqslant (1963/12)\,\mathrm{Re}\{1-b_1\}, \quad |\mathrm{Re}\, b_5| \leqslant (6421/10)\,\mathrm{Re}\{1-b_1\}.$$

The first inequality is not as good as the sharp inequality $|\mathrm{Re}\, b_2| \leqslant 2\,\mathrm{Re}\{1-b_1\}$ proved by Garabedian and Schiffer [2]. They also proved that $\mathrm{Re}\{b_3 - 3b_1\} \geqslant -3$. Together with the inequality $\mathrm{Re}\{b_3 + 3b_1\} \leqslant 3$, which is a special case of what is proved in [4, Theorem 3.3], this implies that $|\mathrm{Re}\, b_3| \leqslant 3\,\mathrm{Re}\{1-b_1\}$. However, the inequalities for the higher coefficients are the only ones of this form known to the authors. For example, they imply

$$\mathrm{Re}\{(1963/12)b_1 - b_4\} \leqslant 1963/12 \quad \text{and} \quad \mathrm{Re}\{(6421/10)b_1 - b_5\} \leqslant 6421/10.$$

It is desirable to have an estimate for (7) which is of a simpler form. Each factor in $\Gamma_{j_1j_2\cdots j_m}$ is dominated by the corresponding factor in $C_{j_1j_2\cdots j_m}(k)$; hence $\Gamma_{j_1j_2\cdots j_m}/C_{j_1j_2\cdots j_m}(k) \leqslant 1$. In addition, each $j_k \leqslant n-m+2$. Therefore

$$\lambda \leqslant \Sigma m^2 2^{m-3}(n-m+2)(n-m+3),$$

where the summation is over all positive integers $j_1, j_2, \ldots, j_m$ with $j_m \geqslant 2$ and $j_1 + j_2 + \cdots + j_m = n + 1$. For fixed m there are $\binom{n-1}{m-1}$ such terms; thus

$$\lambda \leqslant \sum_{m=1}^{n} m^2 2^{m-3}(n - m + 2)(n - m + 3)\binom{n-1}{m-1}$$
$$= (1/4)(4n^4 + 50n^3 + 149n^2 + 277n + 6)3^{n-5} \equiv \mu_n.$$

As a consequence, with μ_n defined just above, our development gives the weaker, but very explicit inequalities

$$|\operatorname{Re} b_n| \leqslant \mu_n \operatorname{Re}\{1 - b_1\} \quad \text{and} \quad \operatorname{Re}\{\mu_n b_1 - b_n\} \leqslant \mu_n.$$

Concluding remark. Just as in Pederson's article [5], we can obtain the following more general theorem.

THEOREM. *Let P be a polynomial of n variables with real coefficients, and assume that $g(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}$ belongs to Σ. Then there exists a constant C, independent of g, such that*

$$|\operatorname{Re}\{P(b_1, b_2, \ldots, b_n) - P(1, 0, \ldots, 0)\}| \leqslant C \operatorname{Re}\{1 - b_1\}.$$

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