HIGH ORDER COEFFICIENT ESTIMATES
IN THE CLASS Σ

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Abstract. Estimates are given for the coefficients $b_n$ of functions in the class Σ in terms of $\text{Re} \{ 1 - b_1 \}$. As a consequence, there is an explicit finite number $\lambda$ such that $\text{Re} \{ \lambda b_1 - b_n \} \leq \lambda$.

Introduction. Let Σ be the class of functions $g(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}$ that are analytic and univalent for $|z| > 1$. The nonvanishing subclass $\Sigma'$ consists of those functions $g \in \Sigma$ for which $g(z) \neq 0$, $|z| > 1$. Denote by $S$ the familiar class of univalent analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the open unit disk. It is easy to see that $f$ belongs to $S$ if and only if $g(z) = 1/f(1/z)$ belongs to $\Sigma'$.

The function $\Psi(z) = z + 1/z$ is extremal for many problems in $\Sigma$ and is closely related to the Koebe function $K(z) = z(1 - z)^{-2}$ in $S$. Its transform

$$k(z^{(n+1)/2})^{2A(n+1)} = z(1 + z^{-n-1})^{2A(n+1)}$$

is extremal for the coefficient problem $\max_{\Sigma} \text{Re} b_n$ for $n = 1$ and $n = 2$, but not for $n > 3$.

It was conjectured by W. E. Kirwan [3] that

$$(1) \quad \text{Re} \{ nb_1 - b_n \} \leq n \quad (n \geq 2)$$

is true for all functions in the class $\Sigma$. Equality always occurs for the function $k$. This conjecture is known (Garabedian and Schiffer [2]) to be true for $n = 2$ and $n = 3$.

In general, we shall consider the inequality

$$(2) \quad \text{Re} \{ \lambda b_1 - b_n \} \leq \lambda.$$  

Our principal result is that this inequality holds in $\Sigma$ for some finite positive $\lambda$ depending on $n$, but not on $g$. If $\lambda' > \lambda$, then

$$\text{Re} \{ \lambda'b_1 - b_n \} = \text{Re} \{ \lambda b_1 - b_n \} + (\lambda' - \lambda) \text{Re} b_1 \leq \lambda + (\lambda' - \lambda) = \lambda';$$

that is, if (2) holds for some $\lambda$, then it holds for all $\lambda' > \lambda$ and equality with $\lambda'$ occurs only for translations of the function $k$. The problem then becomes one of finding the least value $\lambda$ for which (2) is true. It follows from the work in [3] that $\lambda = n$ is the least value for which inequality (2) holds when $n = 2$ or $n = 3$. We refer the reader to the articles [6,7] for further facts related to conjecture (1) and problem (2).
The inequality (2) is contained in the following theorem.

**Theorem.** If \( g(z) = z + \sum_{n=0}^{\infty} b_n z^{-n} \) belongs to \( \Sigma \) and \( n \geq 2 \), then there exists a finite positive number \( \lambda \) such that

\[
|\text{Re} b_n| \leq \lambda \text{Re} \{ 1 - b_1 \}.
\]

In particular, we have \( \text{Re} \{ \lambda b_1 - b_n \} \leq \lambda \).

The number \( \lambda \) is given explicitly in formula (7). Afterwards, there is a section containing the numbers \( \lambda \) for \( n = 2, 3, 4, 5 \) and also simple estimates for \( \lambda \) that depend on \( n \).

Note that (3) may be interpreted as coefficient estimates in a neighborhood of the function \( k \). A generalization is stated as a concluding remark.

**Proof of the theorem.** Our proof is similar, but not identical, to that of R. N. Pederson [5] for a corresponding result in the class \( S \). In addition, we shall keep track of the constants involved.

To estimate the coefficients of \( g \in \Sigma \), we can assume without loss of generality that \( g \) is a slit mapping since such mappings are dense in \( \Sigma \). If \( w_0 \) is any value omitted by \( g \), then \( g - w_0 \) belongs to \( \Sigma' \) and we may associate with its reciprocal a function \( f \in S \), as in the introduction. The function \( f \) may be embedded (cf. [1, Chapter 3]) into a Loewner chain \( f(z, t) \) which satisfies a differential equation of the form

\[
\frac{df}{dt} = z \left[ (1 + k z) / (1 - k z) \right] \frac{df}{dz}, \quad t > 0,
\]

where \( k \) is a piecewise continuous function of \( t \), \( |k| = 1 \), and \( f(z, 0) = f(z) \). In terms of \( g(z, t) = 1/f(z, t) \) the differential equation has the similar form

\[
\frac{dg}{dt} = z \left[ (1 + \bar{k} z) / (1 - \bar{k} z) \right] \frac{dg}{dz}, \quad t > 0,
\]

where \( g(z, 0) = g(z) - w_0 \).

From (4) the coefficients of \( g(z, t) = e^{-t} \left[ z + \sum_{n=0}^{\infty} b_n(t) z^{-n} \right] \) satisfy the differential equation

\[
b'_n(t) = (n + 1) b_n(t) + 2(n - 1) k(t) b_{n-1}(t) + \cdots + 2k^{n-1}(t) b_1(t) - 2k^n(t),
\]

at least for \( n \geq 1 \). Repeated integration gives an iterated integral representation for the coefficients \( b_n = b_n(0), n \geq 1 \), of \( g - w_0 \), hence of \( g \), in the following form:

\[
b_n = \Sigma (-1)^{m+1} 2^m \Gamma_{j_1j_2\cdots j_m} \int_0^\infty \cdots \int_{t_{m-1}}^\infty \kappa_1^{j_1} \kappa_2^{j_2} \cdots \kappa_m^{j_m} dt_1 \cdots dt_2 dt_{m-1},
\]

where \( \kappa_j = \kappa(t_j) e^{-t_j} \), \( \Gamma_j = 1 \), \( \Gamma_{j_1j_2\cdots j_m} = (j_2 + \cdots + j_m - 1)(j_3 + \cdots + j_m - 1) \cdots (j_m - 1) \), and the summation is over all positive integers \( j_1, j_2, \ldots, j_m \) with \( j_m \geq 2 \) and \( j_1 + j_2 + \cdots + j_m = n + 1 \). Of course, \( b_1 \) is simply \( 2 j_0 \kappa(t) e^{-2t} dt \). The coefficient \( b_1 = 1 \) if and only if \( k^2 = 1 \).

**Lemma 1.** For \( j = 1, 2, \ldots, N \), let \( \theta_j \) be real and let \( \delta_j \) be an integer multiple of \( \pi \). Then we have the sharp inequality

\[
|\cos(\theta_1 + \theta_2 + \cdots + \theta_N) - \cos(\delta_1 + \delta_2 + \cdots + \delta_N)| \leq N \left[ \sum_{j=1}^N |\cos \theta_j - \cos \delta_j| \right].
\]
Proof. If \( N = 1 \), the inequality is trivial; assume therefore that \( N \geq 2 \). Let 
\[
\varphi(\theta_1, \ldots, \theta_N) = N^2 - N(\cos \theta_1 + \cdots + \cos \theta_N) + \cos(\theta_1 + \cdots + \theta_N) - 1.
\]
We shall show that \( \varphi \geq 0 \). If some \( \theta_j \) satisfies \( \cos \theta_j \leq 0 \), then 
\[
\varphi \geq N^2 - N(N-1) - 1 - 1 = N - 2 \geq 0.
\]
Thus we may restrict the domain of \( \varphi \) to the cube \( |\theta_j| < \pi/2 \), \( 1 \leq j \leq N \). The critical points of \( \varphi \) are found by setting 
\[
\frac{\partial \varphi}{\partial \theta_j} = N \sin \theta_j - \sin(\theta_1 + \cdots + \theta_N) = 0 \quad \text{for} \quad j = 1, 2, \ldots, N.
\]
That is, \( \sin \theta_j \) has the value \( (1/N)\sin(\theta_1 + \cdots + \theta_N) \) for all \( j \). Since the interval for \( \theta_j \) is restricted to \( (-\pi/2, \pi/2) \), we have \( \theta_1 = \cdots = \theta_N \). Thus at a critical point 
\[
\varphi = N^2 - N^2 \cos \theta + \cos(N\theta) - 1.
\]
This is nonnegative because of the inequality \( 1 - \cos(N\theta) \leq N(1 - \cos \theta) \). We conclude that \( \varphi(\theta_1, \ldots, \theta_N) \geq 0 \) at all points. This proves (6) in case all \( \delta_j = 0 \).

The general case of (6) follows from this special case by replacing each \( \theta_j \) by \( \theta_j + \delta_j \) since \( |\cos(\theta + \delta) - 1| = |\cos \theta - \cos \delta| \) whenever \( \delta \) is an integer multiple of \( \pi \).

Both sides of (6) vanish when all \( \theta_j \) and \( \delta_j \) are zero. To see that the constant \( N \) on the right side cannot be improved, choose all \( \delta_j \) to be zero, set all \( \theta_j \) equal to \( \epsilon \), and send \( \epsilon \) to zero. \( \square \)

Lemma 2. For each real-valued function \( \theta \), there exists a function \( \delta \) whose values are either 0 or \( \pi \) such that 
\[
2|\cos \theta - \cos \delta| \leq [1 - \cos(2\theta)].
\]

Proof. Choose \( \delta \) so that \( \cos \delta = (\cos \theta)/|\cos \theta| \) if \( \cos \theta \neq 0 \) and \( \cos \delta = 1 \) if \( \cos \theta = 0 \). \( \square \)

We represent \( k(t) = e^{i\theta(t)} \) in the differential equation (4) and choose \( \delta(t) \) from now on as in Lemma 2. Then 
\[
4 \int_0^\infty |\cos \theta(t) - \cos \delta(t)| e^{-2t} \, dt \leq 2 \int_0^\infty [1 - \cos 2\theta(t)] e^{-2t} \, dt = \text{Re}\{1 - b_1\}.
\]
We will also use the identity 
\[
2 \int_0^\infty |\cos 2\theta(t) - \cos 2\delta(t)| e^{-2t} \, dt = 2 \int_0^\infty [1 - \cos 2\theta(t)] e^{-2t} \, dt = \text{Re}\{1 - b_1\}.
\]

The choice \( \theta = \delta \) in (5) gives coefficients \( b_n = 0 \) for \( n \geq 2 \). In general, therefore, we have 
\[
\text{Re} \, b_n = \sum (-1)^{m+1} 2^m \Gamma_{j_1 j_2 \cdots j_m} 
\times \int_0^\infty \int_{t_1}^\infty \cdots \int_{t_m}^\infty \left[ \sum \cos \left( \sum_{k=1}^{m} j_k \theta(t_k) \right) - \cos \left( \sum_{k=1}^{m} j_k \delta(t_k) \right) \right] e^{-\Sigma_{k=1}^{m} j_k t_k} \, dt_m \cdots dt_2 \, dt_1.
\]
for \( n \geq 2 \). Let \( I \) denote just the iterated integral. Then Lemma 1 implies

\[
|I| \leq m \sum_{k=1}^{m} \int_{t_1}^{\infty} \cdots \int_{t_{m-1}}^{\infty} |\cos(j_k \theta(t_k)) - \cos(j_k \delta(t_k))| e^{-\Sigma_{j=1}^{m} t_j} dt_m \cdots dt_1
\]

\[
= m \sum_{k=1}^{m} I_k,
\]

where

\[
I_k = \int_{t_1}^{\infty} \cdots \int_{t_{k-1}}^{\infty} |\cos(j_k \theta(t_k)) - \cos(j_k \delta(t_k))| \int_{t_k}^{\infty} \cdots \int_{t_{m-1}}^{\infty} \times e^{-\Sigma_{j=1}^{m} t_j} dt_m \cdots dt_1
\]

\[
= \int_{t_1}^{\infty} \cdots \int_{t_{k-1}}^{\infty} e^{-\Sigma_{j=1}^{k} t_j} \int_{t_k}^{\infty} |\cos(j_k \theta(t_k)) - \cos(j_k \delta(t_k))| \times e^{-\Sigma_{j=1}^{m} t_j} dt_k \cdots dt_1/B_{j_1 \cdots j_m}(k)
\]

and

\[
B_{j_1 \cdots j_m}(m) = 1, \quad B_{j_1 \cdots j_m}(k) = (j_{k+1} + \cdots + j_m)(j_{k+2} + \cdots + j_m) \cdots j_m.
\]

If \( j_k \) is even, then from \( t_k \geq t_{k-1} \) and Lemma 1 we have

\[
\int_{t_{k-1}}^{\infty} |\cos(j_k \theta(t_k)) - \cos(j_k \delta(t_k))| e^{-j_k \theta + \cdots + j_m \delta} dt_k
\]

\[
\leq e^{-j_k \theta + \cdots + j_m \delta} \int_{0}^{\infty} |\cos(j_k \theta(t_k)) - \cos(j_k \delta(t_k))| e^{-2 \theta} dt_k
\]

\[
\leq e^{-j_k \theta + \cdots + j_m \delta} (j_k/2)^2 \int_{0}^{\infty} |\cos(2 \theta(t_k)) - \cos(2 \delta(t_k))| e^{-2 \theta} dt_k
\]

\[
= e^{-j_k \theta + \cdots + j_m \delta} (j_k^2/8) \text{Re} \{1 - b_1\}.
\]

Similarly, if \( j_k \) is odd, then

\[
\int_{t_{k-1}}^{\infty} |\cos(j_k \theta(t_k)) - \cos(j_k \delta(t_k))| e^{-j_k \theta + \cdots + j_m \delta} dt_k
\]

\[
\leq e^{-j_k \theta + \cdots + j_m \delta} \int_{0}^{\infty} |\cos(j_k \theta(t_k)) - \cos(j_k \delta(t_k))| e^{-2 \theta} dt_k
\]

\[
\leq e^{-j_k \theta + \cdots + j_m \delta} ((j_k + 1)/2)
\]

\[
\times \left\{((j_k - 1)/2) \int_{0}^{\infty} |\cos(2 \theta(t_k)) - \cos(2 \delta(t_k))| e^{-2 \theta} dt_k
\right\}
\]

\[
\leq e^{-j_k \theta + \cdots + j_m \delta} ((j_k + 1)/2)
\]

\[
\times \left\{((j_k - 1)/4) \text{Re} \{1 - b_1\} + (1/4) \text{Re} \{1 - b_1\}\right\}
\]

\[
= e^{-j_k \theta + \cdots + j_m \delta} ((j_k + 1)/8) \text{Re} \{1 - b_1\}.
\]
If \(|x|\) denotes the greatest integer less than or equal to \(x\), then we can combine these two cases by writing

\[
\int_{-\infty}^{\infty} |\cos(j_k \theta(i_k)) - \cos(j_k \delta(i_k))| e^{-(j_k + \cdots + j_m) t_k} dt_k 
\leq e^{-(j_k + \cdots + j_m - 2) t_k} \left[ \frac{(j_k + 1)/2}{\Re\{1 - b_1\}} \right].
\]

It follows that

\[
I_k \leq \Re\{1 - b_1\} \left[ \frac{(j_k + 1)/2}{\Re\{1 - b_1\}} \right] \int_{-\infty}^{\infty} \cdots \int_{t_k-3}^{\infty} e^{-\sum_{k=1}^{j_k} t_k} \times e^{-(j_k + \cdots + j_m - 2) t_k} dt_{k-1} \cdots dt_1 \left/ B_{j_1j_2\cdots j_m}(k)\right.
\]

\[
= \Re\{1 - b_1\} \left[ \frac{(j_k + 1)/2}{C_{j_1j_2\cdots j_m}(k)} \right],
\]

where

\[
C_{j_1j_2\cdots j_m}(k) = A_{j_1j_2\cdots j_m}(k) B_{j_1j_2\cdots j_m}(k),
\]

and

\[
A_{j_1j_2\cdots j_m}(1) = 1,
\]

\[
A_{j_1j_2\cdots j_m}(k) = (j_1 + \cdots + j_m - 2)(j_2 + \cdots + j_m - 2) \cdots (j_{k-1} + \cdots + j_m - 2).
\]

In summary, we have the estimate \(\Re b_n \leq \lambda \Re\{1 - b_1\}\), where

\[
\lambda = \sum m 2^{m-2} \Gamma_{j_1j_2\cdots j_m} \sum_{k=1}^{m} j_k \left[ \frac{(j_k + 1)/2}{C_{j_1j_2\cdots j_m}(k)} \right]
\]

and the first summation is over all positive integers \(j_1, j_2, \ldots, j_m\) with \(j_m \geq 2\) and \(j_1 + j_2 + \cdots + j_m = n + 1\). Therefore (3) is proved and also

\[
\Re\{\lambda b_1 - b_n\} \leq \lambda \Re b_1 + |\Re b_n| \leq \lambda \Re b_1 + \lambda \Re\{1 - b_1\} = \lambda. \Box
\]

Estimates for \(\lambda\). Direct computation of (7) leads, for example, to the estimates

\[
|\Re b_2| \leq 8 \Re\{1 - b_1\}, \quad |\Re b_3| \leq (115/3) \Re\{1 - b_1\},
\]

\[
|\Re b_4| \leq (1963/12) \Re\{1 - b_1\}, \quad |\Re b_5| \leq (6421/10) \Re\{1 - b_1\}.
\]

The first inequality is not as good as the sharp inequality \(|\Re b_2| \leq 2 \Re\{1 - b_1\}\) proved by Garabedian and Schiffer [2]. They also proved that \(\Re\{b_3 - 3b_1\} \geq -3\). Together with the inequality \(\Re\{b_3 + 3b_1\} \leq 3\), which is a special case of what is proved in [4, Theorem 3.3], this implies that \(|\Re b_3| \leq 3 \Re\{1 - b_1\}\). However, the inequalities for the higher coefficients are the only ones of this form known to the authors. For example, they imply

\[
\Re\{\left(1963/12\right) b_1 - b_4\} \leq 1963/12 \quad \text{and} \quad \Re\{\left(6421/10\right) b_1 - b_5\} \leq 6421/10.
\]

It is desirable to have an estimate for (7) which is of a simpler form. Each factor in \(\Gamma_{j_1j_2\cdots j_m}\) is dominated by the corresponding factor in \(C_{j_1j_2\cdots j_m}(k)\); hence

\[
\Gamma_{j_1j_2\cdots j_m}/C_{j_1j_2\cdots j_m}(k) \leq 1.
\]

In addition, each \(j_k \leq n - m + 2\). Therefore

\[
\lambda \leq \sum m 2^{m-3} (n - m + 2)(n - m + 3),
\]

for all \(n \geq 2\) and \(m \leq n\).
where the summation is over all positive integers \( j_1, j_2, \ldots, j_m \) with \( j_m \geq 2 \) and \( j_1 + j_2 + \cdots + j_m = n + 1 \). For fixed \( m \) there are \( \binom{n+1}{m-1} \) such terms; thus
\[
\lambda \leq \sum_{m=1}^{n} m^22^{m-3}(n-m+2)(n-m+3)\binom{n-1}{m-1}
\]
\[
= \left(\frac{1}{4}\right)(4n^4 + 50n^3 + 149n^2 + 277n + 6)3^{n-5} \equiv \mu_n.
\]
As a consequence, with \( \mu_n \) defined just above, our development gives the weaker, but very explicit inequalities
\[
|\text{Re} b_n| \leq \mu_n \text{Re} \{1 - b_1\} \quad \text{and} \quad \text{Re} \{\mu_n b_1 - b_n\} \leq \mu_n.
\]

Concluding remark. Just as in Pederson's article [5], we can obtain the following more general theorem.

Theorem. Let \( P \) be a polynomial of \( n \) variables with real coefficients, and assume that \( g(z) = z + \sum_{n=0}^{\infty} b_n z^{-n} \) belongs to \( \Sigma \). Then there exists a constant \( C \), independent of \( g \), such that
\[
|\text{Re} \{P(b_1, b_2, \ldots, b_n) - P(1, 0, \ldots, 0)\}| \leq C \text{Re} \{1 - b_1\}.
\]

References
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