THE ASYMPTOTIC-NORMING AND THE RADON-NIKODYM PROPERTIES ARE EQUIVALENT IN SEPARABLE BANACH SPACES

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ABSTRACT. We show that the asymptotic-norming and the Radon-Nikodym properties are equivalent, settling a problem of James and Ho [9]. In the process, we give a positive solution to two questions of Edgar and Wheeler [6] concerning Čech-complete Banach spaces. We also show that a separable Banach space with the Radon-Nikodym property semi-embeds in a separable dual whenever it has a norming space not containing an isomorphic copy of $l_1$. This gives a partial answer to a problem of Bourgain and Rosenthal [3].

Introduction. Let $X$ be a Banach space. We recall that $X$ is said to have the point of continuity property (resp. the Radon-Nikodym property) if every weakly closed bounded subset of $X$ has a point of weak to norm continuity (resp. a denting point). A separable Banach space $X$ is said to have the asymptotic norming property if there exists a separable Banach space $Y$ such that $X$ is (isomorphic to) a subspace of $Y^*$ which verifies the following property:

\[
\text{(A.N.P.)} \quad \text{if } (x_n) \subseteq X, x_n \rightharpoonup y^* \text{ and } \|x_n\| \to ||y^*||, \\
\text{then } \lim_{n} \|x_n - y^*\| = 0.
\]

In [9], James and Ho introduced the asymptotic norming property, proved that it implies the Radon-Nikodym property and asked whether the two properties are equivalent. To prove this conjecture we recall that Davis and Johnson [5] showed that for every separable subspace $X$ of $Y^*$, the latter can be renormed in such a way that the conclusion of (A.N.P.) holds provided $y^*$ is assumed to be in $X$. The only missing ingredient in the equivalent norm is then the term that forces $y^*$ to be in $X$. On the other hand, the authors proved in [7] the following

**Theorem 1 [7].** Let $X$ be a separable Banach space. Then $X$ has the point of continuity property (resp. the Radon-Nikodym property) if and only if $X$ embeds isometrically in the dual of a separable Banach subspace $Y$ of $X^*$ in such a way that $Y^* \setminus X = \bigcup_n K_n$ where each $K_n$ is weak*-compact (resp. weak*-compact and convex) in $Y^*$.

It is easy to see that if the convex $K_n$’s can be chosen to be a strictly positive distance away from $X$, then the distance of the elements of $Y^*$ to the $K_n$’s (made into a suitable seminorm) would give the missing ingredient that forces $y^*$ to be in $X$. 

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The problem of the existence of such \( K_n \)'s coincide with a question of Edgar and Wheeler \([6]\) on strongly Čech-complete Banach spaces. The following theorem gives a positive answer to these questions.

**Theorem (1) Bis.** Let \( X \) be a separable Banach space. Then \( X \) has the point of continuity property (resp. the Radon-Nikodym property) if and only if \( X \) embeds in the dual of a separable Banach space \( Y \) in such a way that \( Y^* \setminus X = \bigcup_n K_n \) where each \( K_n \) is weak\(^*-\)compact (resp. weak\(^*-\)compact and convex) satisfying \( d(K_n, X) > 0 \).

The proof will be broken into several lemmas. We shall need the following notations and terminology: If \( C \) is a subset of a dual space \( Y^* \) we shall denote by \( \overline{C}^w \) (resp. \( \overline{C} \)) its weak\(^*-\)closure (resp. its norm closure). The distance between two subsets \( C \) and \( D \) of \( Y^* \) will be denoted by \( d(C, D) = \inf \{\|x - y\|; x \in C, y \in D\} \).

If \( \Delta \) is any metric on \( Y^* \), \( l \in Y^* \) and \( \rho > 0 \), then \( B_{\Delta}(l, \rho) \) will be the \( \Delta \)-open ball \( \{y \in Y; \Delta(y, l) < \rho\} \). If \( \Delta \) is induced by the norm we shall simply write \( B(l, \rho) \). The closed unit ball of a Banach space \( Z \) will be denoted \( B_Z \).

If now \( X \) is a subspace of \( Y^* \) and \( L \) is a \( w^*-\)compact subset of \( Y^* \) which is disjoint from \( X \), we shall say that \( L \) is \( \rho \)-bad with respect to \( X \) for some \( \rho > 0 \) if

1. for each \( \varepsilon > 0 \), the set \( L_\varepsilon = \{l \in L; d(l, X) < \varepsilon\} \) is \( w^* \)-dense in \( L \);
2. the set \( l^\rho = \{l \in L; d(l, X) \geq \rho\} \) is also \( w^* \)-dense in \( L \).

**Lemma (1).** Let \( X \) be a subspace of the dual of a separable Banach space \( Y \), such that \( B_X^* = B_Y^* \). Let \( L \) be a \( w^* \)-compact subset of \( \theta \cdot B_Y^* \) \( (\theta < 1) \) which is \( \rho \)-bad with respect to \( X \) for some \( \rho > 0 \). Let \( K \) be a \( w^* \)-compact subset of \( B_Y^* \) which is disjoint from \( X \). Then for each \( \varepsilon > 0 \) such that \( 0 < \varepsilon < 1 - \theta \) and each \( w^* \)-open subset \( V \) of \( B_Y^* \) with \( V \cap L \neq \emptyset \), there exist \( l \in L, x \in X \) such that

1. \( ||x|| \leq \varepsilon \),
2. \( l + x \in V \),
3. \( l + x \notin K \),
4. \( d(l + x, X) \geq \rho - \varepsilon \).

**Proof.** Choose \( l_1 \in L \) such that \( l_1 \in V \) and \( d(l_1, X) < \varepsilon/2 \). Consider \( x_1 \) and then \( (y_n) \in X \) such that \( l_1 = w^*\lim_n (x_1 + y_n) \) with \( ||y_n|| \leq \varepsilon/2 \). Note that \( ||x_1|| \leq \theta + \varepsilon/2 \), hence \( x_1 + y_n \in B_X \). Choose now \( n_0 \) such that \( x_1 + y_{n_0} \in V \). We have that \( x_1 + y_{n_0} \notin K \). On the other hand, \( w^*\lim_n (l_1 - y_n + y_{n_0}) = x_1 + y_{n_0} \) and \( ||l_1 - y_n + y_{n_0}|| \leq 1 \), hence, for a large enough \( n_1 \), we have \( (l_1 - y_{n_1} + y_{n_0}) \in V \) and \( (l_1 - y_n + y_{n_0}) \notin K \).

Since \( L \) is \( \rho \)-bad, choose \( (l_m) \subseteq L \), \( d(l_m, X) \geq \rho \) such that \( l_1 = w^*\lim_m (l_m) \). That is, for a large enough \( m_0 \), we have \( (l_m - y_{n_1} + y_{n_0}) \in V \) and \( (l_m - y_{n_1} + y_{n_0}) \notin K \). Now take \( l = l_{m_0} \) and \( x = (-y_{n_1} + y_{n_0}) \). They clearly verify the claimed properties.

**Lemma (2).** Let \( X \) be a separable subspace of the dual of a separable Banach space \( Y \) such that \( B_X \) is a \( w^* \)-Gs, \( w^* \)-dense in \( B_Y^* \). Then for each \( \theta < 1, \theta B_Y \) contains no \( \rho \)-bad \( w^* \)-compact sets with respect to \( X \) for any \( \rho > 0 \).

**Proof.** Write \( B_Y \setminus B_X = \bigcup_n K_n \) where \( (K_n) \) is an increasing sequence of \( w^* \)-compact sets. Let \( \Delta \) be a distance defining the \( w^* \)-topology on \( B_Y \) and let \( (z_n) \) be a dense sequence in \( B_X \). Suppose \( L \) is a \( w^* \)-compact subset of \( \theta \cdot B_Y \) \( (\theta < 1) \) which is \( \rho \)-bad with respect to \( X \) for some \( \rho > 0 \).
Let \( l_0 \) be any point in \( L \) and let \( V_0 = B_\Delta(l_0,1) \) and \( 0 < \varepsilon_0 < \inf(1 - \theta, \rho/2) \). Use Lemma (1) to obtain \( x_0 \in X \), \( \|x_0\| \leq \varepsilon_0 \) and \( l'_0 \in L \) such that \( l'_0 + x_0 \in V_0 \), \( l'_0 + x_0 \notin K_0 \) and \( d(l'_0 + x_0, X) \geq \rho - \varepsilon_0 > \rho/2 \).

Now set \( L_1 = L + x_0, \theta_1 = \theta + \varepsilon_0 < 1 \) and \( l_1 = l'_0 + x_0 \).

Note that \( L_1 \) is also \( \rho \)-bad with respect to \( X \) and \( L_1 \subseteq \theta_1 \cdot B_Y \). Let \( V'_1 \) be a \( w^* \)-open subset of \( V_0 \) containing \( l_1 \) such that \( V'_1 \cap (K_0 \cup B^*(z_0, \rho/2)) = \emptyset \). Set \( V_1 = V'_1 \cap B_\Delta(l_1,1/2) \) and \( \varepsilon_1 < \inf(1 - \theta_1, \rho/2) \) and apply again Lemma 1 to obtain \( x_1 \in X, \|x_1\| = \varepsilon_1, l'_1 \in L_1, \ l'_1 + x_1 \in V_1, \ l'_1 + x_1 \notin K_1 \) and \( d(l'_1 + x_1, X) \geq \rho - \varepsilon_1 > \rho/2 \).

By induction, we get a decreasing sequence \((V_n)\) of \( w^* \)-open subsets of \( B_Y \) and a sequence \((l_n)\) of vectors such that

(i) \( l_n \in V_n \) for each \( n \),

(ii) \( \text{diam}_A(V_n^*) \leq 2^{-n} \),

(iii) \( V_n^* \cap (K_{n-1} \cup (\bigcup_{j=0}^{n-1} B^*(z_j, \rho/2))) = \emptyset \).

It follows that the \( w^* \)-limit \( l_\infty \) of \((l_n)\) can neither be in \( B_X \) nor in any of the \( K_n \)'s, which is obviously a contradiction since \( l_\infty \in B_Y \).

**Lemma (3).** Let \( X \) be a separable subspace of the dual of a separable Banach space \( Y \) such that \( B_X \) is a \( w^* \)-Gδ, \( w^* \)-dense in \( B_Y \). Let \( L \) be a subset of \( Y^* \) which is disjoint of \( X \). Then:

(i) If \( L \) is \( w^* \)-compact, there exists a \( w^* \)-open set \( V \) such that \( L \cap V \neq \emptyset \) and \( d(L \cap V^*, X) > 0 \).

(ii) If \( L \) is \( w^* \)-compact and convex, there exists a \( w^* \)-open half-space \( V \) such that \( L \cap V \neq \emptyset \) and \( d(L \cap V^*, X) > 0 \).

**Proof.** We first claim that there exists \( \varepsilon > 0 \) such that the set \( L_\varepsilon = \{ l \in L; d(l, X) \leq \varepsilon \} \) is not \( w^* \)-dense in \( L \). Indeed, suppose not. We can assume without loss that \( L \subseteq B_Y/2 \). Now note that \( L = \bigcup_n L^n \) where \( L^n = \{ l; d(l, X) \geq 1/n \} \) since \( L \cap X = \emptyset \). It follows that there exists \( m \) such that \( L^m \) has a nonempty interior \( V_0 \) in the \( w^* \)-topology relative to \( L \). It follows that \( V_0^* \) is a 1/m-bad set with respect to \( X \) which clearly contradicts Lemma (2).

In case (i) we take \( V \) to be a \( w^* \)-open subset of \( Y^* \) such that \( V \cap L = L \setminus L_\varepsilon^* \) which is nonempty.

In case (ii), note that \( L_\varepsilon^* \) is also convex, hence, any \( w^* \)-open half-space \( V \) that separates any point \( l \) in \( L \setminus L_\varepsilon^* \) from \( L_\varepsilon^* \) will do the job.

**Lemma (4).** Let \( X \) be a separable subspace of the dual of a separable Banach space \( Y \) such that \( B_X \) is a \( w^* \)-Gδ set which is \( w^* \)-dense in \( B_Y \). Let \( L \) be a subset of \( Y^* \) which is disjoint of \( X \). Then:

(i) If \( L \) is \( w^* \)-compact, there exists a countable collection of \( w^* \)-compact sets \((L_n)\) whose union is \( L \) such that \( d(L_n, X) > 0 \) for each \( n \).

(ii) If \( L \) is \( w^* \)-compact and convex, there exists a countable collection of \( w^* \)-compact convex sets whose union is \( L \) such that \( d(L_n, X) > 0 \) for each \( n \).

**Proof.** (i) By transfinite induction, we define a decreasing family \((K_\alpha)\) of \( w^* \)-compact subsets of \( L \) in the following manner:

(a) \( K_0 = L \).

(b) If \( \alpha = \beta + 1 \) and \( K_\beta \) nonempty, apply Lemma (3) to \( K_\beta \) to obtain a \( w^* \)-open set \( V_\beta \) such that \( K_\beta \cap V_\beta \neq \emptyset \) and \( d(K_\beta \cap V_\beta^*, X) > 0 \). Set \( K_\alpha = K_\beta \setminus V_\beta \).
(c) If \( \alpha \) is a limit ordinal, set \( K_\alpha = \bigcap_{\beta < \alpha} K_\beta \).

Since \( L \) is \( w^* \)-metrizable there exists \( \gamma < \Omega \) (the first uncountable ordinal) such that \( K_\gamma = \emptyset \). It is clear that \( L = \bigcup_{\alpha < \gamma} K_\alpha \cap V_\alpha^* \) and \( L_\alpha = K_\alpha \cap \overline{V_\alpha^*} \) is a strictly positive distance away from \( X \) for each \( \alpha < \gamma \).

(ii) If \( L \) is also convex, then \( V \) can be taken to be a \( w^* \)-open half-space by Lemma (3), hence each \( L_\alpha = K_\alpha \cap \overline{V_\alpha^*} \) is then \( w^* \)-compact and convex.

The following is now immediate:

**Theorem (1) Ter.** Let \( X \) be a separable subspace of the dual of a separable Banach space \( Y \) such that \( B_X \) is \( w^* \)-dense in \( B_Y^* \). If \( Y^* \setminus X = \bigcup_n K_n \) where each \( K_n \) is \( w^* \)-compact (resp. \( w^* \)-compact and convex), then \( Y^* \setminus X = \bigcup_n K'_n \) where each \( K'_n \) is \( w^* \)-compact (resp. \( w^* \)-compact and convex) such that \( d(K'_n, X) > 0 \).

**Proof of Theorem (1) Bis.** If \( X \) is a separable Banach space with the point of continuity property, apply Theorem (1) to get a separable Banach subspace \( Y \) of \( X^* \) such that \( X \) is a subspace of \( Y^* \) verifying \( Y^* \setminus X = \bigcup_n K_n \) where each \( K_n \) is \( w^* \)-compact. It follows that \( B_X \) is a \( w^* \)-open, \( w^* \)-dense subset of \( B_Y^* \). Apply now Theorem (1) ter to get the conclusion.

If \( X \) has the Radon-Nikodym property, each \( K_n \) is then convex, and Theorem (1) ter applies again and gives the claimed result.

The following corollary answers two questions of Edgar and Wheeler [6]:

**Corollary (5).** (a) A separable Banach space \( X \) has the point of continuity property and its dual \( X^* \) is separable if and only if \( X^{**} \setminus X \) is the countable union of \( w^* \)-compact sets \( (K_n) \) such that \( d(K_n, X) > 0 \).

(b) A separable Banach space \( X \) has the Radon-Nikodym property and its dual \( X^* \) is separable if and only if \( X^{**} \setminus X \) is the countable union of \( w^* \)-compact convex sets \( (K_n) \) such that \( d(K_n, X) > 0 \).

**Proof.** In view of the results of [6 and 7] the space \( Y \) mentioned in Theorem (1) can be taken in this case to be the separable dual \( X^* \).

Theorem (1) bis and the proof of Theorem 4.14 of [6] applied to \( Y \) instead of \( X^* \) gives the following

**Corollary (6).** A separable Banach space \( X \) has the point of continuity property if and only if there exists a separable Banach space \( Y \) and a family of norm one vectors \( \{y_{n,i}; 1 \leq i \leq m_n, n \in \mathbb{N}\} \) in \( Y \) such that

\[
X = \left\{ y^* \in Y^*; \lim_{n} \max_{1 \leq i \leq m_n} |y^*(y_{n,i})| = 0 \right\}.
\]

The following settles a question of James and Ho [9]:

**Theorem (2).** A separable Banach space \( X \) has the Radon-Nikodym property if and only if it has the asymptotic-norming property.

**Proof.** Suppose that \( X \) has the Radon-Nikodym property. Apply Theorem (1) bis to obtain a separable Banach space \( Y \) such that \( X \) is a subspace of \( Y^* \) verifying \( Y^* \setminus X = \bigcup_n K_n \) where each \( K_n \) is \( w^* \)-compact convex and \( d(K_n, X) \geq \varepsilon_n > 0 \). Following Davis and Johnson [5], let \( (E_n)_n \) be an increasing sequence of
finite-dimensional subspaces of $X$ such that $X = \bigcup_n E_n$ and define the seminorm $||x|| = \sum_n 2^{-n}d(x, E_n)$. Now let $!_n$ be the seminorm defined by

$$!_n = d(x, R+K_n) + d(x, -R+K_n)$$

and set $! = \sum_n 2^{-n}!_n$. Finally, let $||x||_1 = ||x|| + |||x||| + !$. Note that $||x|| \leq ||x||_1 \leq 7||x||$ for each $x$ in $Y^*$ and that $||_1$ is $w^*$-lower semicontinuous, hence, it is a dual norm on $Y^*$.

Suppose now that $(x_n) \subseteq X$, $y^* \in Y^*$ such that $||x_n||_1 \to ||y^*||_1$ and $w^*$-$\lim_n (x_n) = y^*$. Since each piece of the norm is $w^*$-lower semicontinuous we get that $||x_n|| \to ||y^*||$, $|||x_n||| \to ||y^*||$ and $d(x_n, R+K_m) \to d(y^*, R+K_m)$ for each $m$.

We claim that $y^* \in X$. Indeed, if not, then there exists an $m$ such that $y^* \in K_m$ and $\lim n d(x_n, R+K_m) = d(y^*, R+K_m) = 0$.

We can then suppose that $||x_n - \lambda_n k_n|| \leq 1/n$ for some $\lambda_n \in R_+$ and $k_n \in K_m$. This gives

$$1/n \geq ||x_n - \lambda_n k_n|| = \lambda_n ||x_n/\lambda_n - k_n|| \geq \lambda_n \varepsilon_m.$$ 

It follows that $\lambda_n \to 0$ and $||x_n|| \to 0$, a contradiction. Since $y^*$ is now in $X$, the Davis-Johnson norm insures that $\lim_n ||x_n - y^*|| = 0$.

The converse was proved by James and Ho [9]. We sketch an easier proof based on martingales and already used by Davis et al. [4]. Let $D$ be a countable dense set in the unit ball of $Y$. Let $(\phi_n)$ be an $X$-valued bounded martingale. Let $\phi_\infty$ be a $w^*$-limit of $(\phi_n)$ which is valued in $Y^*$. For each $y \in D$, the real-valued martingale $y(\phi_n)$ converges to $y(\phi_\infty)$ outside a set $\Omega_y$ of measure zero.

By a lemma of Neveu [11], the martingale $||\phi_n|| = \sup_{y \in D} |y(\phi_n)|$ converges to $\sup_{y \in D} |y(\phi_\infty)| = ||\phi_\infty||$ outside a set $\Omega_0$ of measure zero. Since $X$ has the asymptotic norming property with respect to $Y$, we get that $\lim_n ||\phi_n - \phi_\infty|| = 0$ outside the set $\Omega_0 \cup \bigcup_{y \in D} \Omega_y$ which is of measure zero.

Recall that a bounded linear operator $T$ from a Banach space $X$ into a space $Y$ is said to be a semi-embedding if it is one-to-one and if the image of the unit ball of $X$ by $T$ is norm closed in $Y$. In [3], Bourgain and Rosenthal showed that the $L_\infty$-spaces constructed by Bourgain and Delbaen [2] do not semi-embed in separable duals even though they enjoy the Radon-Nikodym property. On the other hand, they show that the Radon-Nikodym spaces constructed by Johnson and Lindenstrauss [10] do semi-embed in separable duals even though they do not embed in such spaces. The following theorem gives a sufficient condition that guarantees such semi-embeddings for Radon-Nikodym spaces. It gives a partial solution to a question of Bourgain and Rosenthal [3].

**Theorem (3).** If $X$ is a separable Banach space with a norming space not containing an isomorphic copy of $l_1$, then $X$ has the Radon-Nikodym property if and only if it semi-embeds in a separable dual.

First we need the following

**Lemma (5).** Let $Y$ be a separable Banach space not containing an isomorphic copy of $l_1$. If $X$ is a separable subspace of $Y^*$ with the Radon-Nikodym property such that $B_X$ is $w^*$-dense in $B_Y^*$, then the orthogonal of $X$ in $Y^{**}$ is $w^*$-separable.

**Proof.** By a theorem of Bourgain [1], $B_X$ is then $w^*$-dentable in $Y^*$; that is, every norm closed convex subset of $B_X$ contains $w^*$-open slices with arbitrarily
small diameters. Now, we proceed as in Lemma III.1 of [7]: Fix $\varepsilon > 0$ and define inductively a decreasing family of norm-closed convex subsets $(F_\alpha)$ of $B_X$ in the following way:

(i) $F_0 = B_X$.

(ii) If $\alpha = \beta + 1$ and $F_\beta \neq 0$, use the $w^*$-dentability to find a $w^*$-open slice $S_\beta$ of $F_\beta$ such that $\text{diam}(S_\beta) < \varepsilon$. Set $F_\alpha = F_\beta \setminus S_\beta$.

(iii) If $\alpha$ is a limit ordinal, let $F_\alpha = \bigcap_{\beta < \alpha} F_\beta$.

Since $B_X$ is separable, there exists $\gamma < \Omega$ (the first uncountable ordinal) such that $F_\gamma = \emptyset$ and $F_\beta \neq 0$ for $\beta < \gamma$. Let $K_\alpha$ be the $w^*$-closure of $F_\alpha$ in $Y^*$ and let $H_\alpha$ be the $w^*$-open half-space such that $S_\alpha = H_\alpha \cap F_\alpha$. It is clear that

$$B_X \subseteq \bigcap_{\alpha \leq \gamma} \left( K_\alpha \cup \bigcup_{\beta < \alpha} H_\beta \right).$$

Moreover, if $x$ belongs to the set on the right-hand side, then $x \in K_\beta \cap H_\beta$ for some $\beta < \gamma$ which implies that $d(x, B_X) \leq \varepsilon$. It follows that if we repeat the construction for each $\varepsilon = 1/n$ we then get

$$B_X = \bigcap_{n} \bigcap_{\alpha \leq \gamma_n} \left( K_{\alpha, n} \cup \bigcup_{\beta < \alpha} H_{\beta, n} \right).$$

Since $Y$ is separable, write that $K_{\alpha, n} = \bigcap_m L_{\alpha, n, m}$ where each $L_{\alpha, n, m}$ is a $w^*$-open half-space in $Y^*$. It follows that $Y^* \setminus B_X$ and hence $Y^* \setminus X$ is a countable union of $w^*$-compact convex subsets $(K_\alpha)$ of $Y^*$. By Theorem (1) ter, we can suppose that $d(K_\alpha, X) > \varepsilon_n > 0$. If $\pi$ is now the quotient map from $Y^*$ onto $Y^*/X$, we obtain that $0 \in \pi(K_\alpha)$ for each $n$, hence, there exists $f_n$ in $(Y^*/X)^* = X^\perp$ such that $f_n > \varepsilon_n$ on $\pi(K_\alpha)$. It is now clear that $X = \{y^* \in Y^*; f_n(y^*) = 0; \forall n \in \mathbb{N}\}$ and that $X^\perp$ is $w^*$-separable.

**Proof of Theorem (3).** Since $l_1$ does not embed in $Y$, use Odell and Rosenthal's theorem [12] to find for each $n$ a sequence $(g_{n,m})$ in $Y$ that converges pointwise on $Y^*$ to $f_n$. The space $X$ can now be written as $\{y^* \in Y^*; \lim_{m \to \infty} g_{n,m}(y^*) = 0; \forall n \in \mathbb{N}\}$. We may suppose that $\|g_{n,m}\| \leq 1$ for each $n, m \in \mathbb{N}$. Define now for each $n \geq 1$, the operator $T_n : l_1 \to Y$ by $T_n(\alpha_m)_m = \sum_n \alpha_m g_{n,m}$ and let $T_0 : l_2 \to Y$ be a dense range operator. Let $T : l_2 \oplus (\sum_n l_1)_l \to Y$ be the unique linear operator whose restrictions to the factor spaces are those given by the sequence $\{T_0, (2^{-n}T_n)_{n \geq 1}\}$. Let $T^* : Y^* \to l_2 \oplus (\sum_n l_\infty)_l$ be the dual operator which is one-to-one since $T_0^*$ is. We claim that $T^*$ is valued in $l_2 \oplus (\sum_n c)_l$ and that $T^*(B_X)$ is norm-closed. Indeed, we have for each $n \geq 1$ and each $y^* \in Y^*$,

$$(T_n^*(y^*)) = (g_{n,m}(y^*))_m$$

which is convergent to $f_n(y^*)$. Moreover, if $x_l \in B_X$ and $\lim_l T^* x_l = z$, then $z = T^* y^*$ where $y^*$ is a $w^*$-limit of $(x_l)$ in $Y^*$ since $T^*$ is one-to-one and $w^*$-to-$w^*$ continuous. Moreover, since for each $n$, $g_{n,m}(x_l) \to g_{n,m}(y^*)$ uniformly in $m$, we get that $f_n(x_l) \to f_n(y^*)$ and $f_n(y^*) = 0$ for each $n$. It follows that $y^* \in B_X$.

The operator $T$ has a separable adjoint, hence the Stegall factorization theorem [13] applies and we get a separable Banach space $Z$ with a separable dual such that $T = U \circ V$ where $U : Z \to Y$ and $V : l_2 \oplus (\sum_n l_1)_l \to Z$. Note now that $U^*(B_X)$ is norm-closed in $Z^*$ since $T^*(B_X)$ is closed in $l_2 \oplus (\sum_n l_\infty)_l$. Hence $U^*$ is a semi-embedding of $X$ into the separable dual $Z^*$.
REMARK. The proof of Theorem (1) bis relies heavily on the fact that the ball of $X$ is a $w^*-G_δ$ in $Y^*$. Actually the local statement is not true. In a forthcoming paper, we construct a $w^*-G_δ$ subset $C$ of a dual space whose complement is not decomposable into a countable union of $w^*$-compact sets which are a strictly positive distance away from $C$. This question is closely related to the problem of minimizing a certain class of functions on the set $C$. We shall deal with these questions in [8].

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