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INEQUALITIES RELATING SECTIONAL CURVATURES OF A SUBMANIFOLD TO THE SIZE OF ITS SECOND FUNDAMENTAL FORM AND APPLICATIONS TO PINCHING THEOREMS FOR SUBMANIFOLDS

RALPH HOWARD AND S. WALTER WEI

ABSTRACT. The Gauss curvature equation is used to prove inequalities relating the sectional curvatures of a submanifold with the corresponding sectional curvature of the ambient manifold and the size of the second fundamental form. These inequalities are then used to show that if a manifold $\overline{M}$ is δ-pinched for some $\delta > \frac{1}{4}$, then any submanifold M of $\overline{M}$ that has small enough second fundamental form is δ_M-pinched for some $\delta_M > \frac{1}{4}$. It then follows from the sphere theorem that the universal covering manifold of M is a sphere. Some related results are also given.

1. Introduction. This note is motivated by questions of the following type: Let $\overline{M}$ be a complete Riemannian manifold and M a compact immersed submanifold of $\overline{M}$; how then is the topology of M affected by placing a sufficiently small upper bound on the size of the second fundamental form of M in $\overline{M}$? For example, when $\overline{M}$ is isometric to a standard sphere, Lawson and Simons [**L-S**] show that if the length of the second fundamental form of M is small enough, then M is a homotopy sphere. If $\overline{M}$ is the product of two spheres, then the second author has shown in [**Wei**] that the submanifolds of $\overline{M}$ with sufficiently small second fundamental are homeomorphic to totally geodesic submanifolds of $\overline{M}$.

Here we will consider the case that $\overline{M}$ is δ-pinched for some $\delta > \frac{1}{4}$. That is, all sectional curvatures of $\overline{M}$ are in the closed interval $[\delta K_0, K_0]$ for some constant $K_0 > 0$. In this case the well-known sphere theorem of Berger, Klingenberg, Rauch and Toponogov implies that the universal covering manifold of $\overline{M}$ is homeomorphic to a sphere. If $\overline{M}$ and M are both simply connected and M has codimension one, then Flaherty has given conditions (cf. §3 below) on the second fundamental form of M which forces M to be a homotopy sphere.

In this note we will extend this to higher codimensions and at the same time weaken the assumptions on the second fundamental form of M and drop the assumption of simple connectivity on $\overline{M}$.

Our method is to use the Gauss curvature equation to prove inequalities relating the sectional curvatures of a submanifold with the corresponding sectional curvatures of the ambient manifold and the size of the second fundamental form of the submanifold. These inequalities then imply that a submanifold of a pinched manifold is also pinched (with a slightly worse pinching constant) provided that its second fundamental form is small enough. The proofs of these inequalities are elementary; they only involve completing the square.

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2. The inequalities. Let M be an n-dimensional ($n \geq 2$) submanifold isometrically immersed in the Riemannian manifold $\overline{M}$. At each point $x \in M$ the tangent space to M at x will be written as TM_x and the normal space to M at x as $T^{\perp}M_x$. The second fundamental form h_x of M in $\overline{M}$ at x is a symmetric bilinear form $TM_x \times TM_x$ to $T^{\perp}M_x$. If $e_1, \dots, e_n$ is any orthonormal basis on TM_x, then the length of h_x is defined by

$$\|h_x\|^2 = \sum_{1 \leq i,j \leq n} \|h_x(e_i, e_j)\|^2. \tag{1}$$

If P is a plane section of M at x, i.e. a two-dimensional subspace of TM_x, then denote by $\overline{K}(P)$ the sectional curvature of $\overline{M}$ at P, by $K(P)$ the sectional curvature of M at P and by $h|_P$ the symmetric bilinear from $P \times P$ to $T^{\perp}M_x$ obtained by restricting h_x to $P \times P$. Let e_1, e_2 be any orthonormal basis of P. Then the Gauss curvature equation can be written as

$$K(P) = \overline{K}(P) + \langle h(e_1, e_1), h(e_2, e_2)\rangle - \|h(e_1, e_2)\|^2 \tag{2}$$

and the length of $h|_P$ is

$$\begin{aligned} \|h|_P\|^2 &= \sum_{1 \leq i,j \leq 2} \|h(e_i, e_j)\|^2 \\ &= \|h(e_1, e_1)\|^2 + 2\|h(e_1, e_2)\|^2 + \|h(e_2, e_2)\|^2. \end{aligned} \tag{3}$$

Clearly $\|h|_P\|^2 \leq \|h_x\|^2$. Our estimates are

PROPOSITION 1. *If P is a plane section of M, then*

$$\begin{aligned} \overline{K}(P) - \tfrac{1}{2}\|h\|^2 \leq \overline{K}(P) - \tfrac{1}{2}\|h|_P\|^2 &\leq K(P) \\ &\leq \overline{K}(P) + \tfrac{1}{2}\|h|_P\|^2 \leq \overline{K}(P) + \tfrac{1}{2}\|h\|^2. \end{aligned}$$

PROPOSITION 2. *If M is a minimal surface in $\overline{M}$, then*

$$\overline{K}(P) - \tfrac{1}{2}\|h\|^2 = K(P) \leq \overline{K}(P).$$

PROPOSITION 3. *If M is a totally umbilic surface in $\overline{M}$, then*

$$\overline{K}(P) \leq K(P) = \overline{K}(P) + \tfrac{1}{2}\|h\|^2.$$

PROPOSITION 4. *If $\overline{M}$ is a Kaehler manifold and M is a Kaehler submanifold of $\overline{M}$, then for every holomorphic plane section P of M*

$$\overline{K}(P) - \tfrac{1}{2}\|h\|^2 \leq \overline{K}(P) - \tfrac{1}{2}\|h|_P\|^2 = K(P) \leq \overline{K}(P).$$

REMARKS. Propositions 2 and 3 show that the inequalities in Proposition 1 are sharp in the case that M is two-dimensional. By considering cylinders over minimal surfaces or umbilic surfaces in Euclidean space it is possible to show that the inequalities in Proposition 1 are sharp in all dimensions. Proposition 4 is a restatement of Proposition 9.2 in Volume 2 of [**K-N**]. It is included here because of its relation to the other results.

PROOF. Let e_1, e_2 be an orthonormal basis of P. Let $X = h(e_1, e_1)$, $Y = h(e_1, e_2)$ and $Z = h(e_2, e_2)$. Because of equations (2) and (3), to prove Proposition 1 it is enough to show that

$$-(\|X\|^2 + 2\|Y\|^2 + \|Z\|^2) \leq 2(\langle X, Z\rangle - \|Y\|^2) \leq \|X\|^2 + 2\|Y\|^2 + \|Z\|^2.$$

This follows at once from the identities

$$\|x\|^2 + 2\|Y\|^2 + \|Z\|^2 - 2(\langle X, Z\rangle - \|Y\|^2) = \|X - Z\|^2 + 4\|Y\|^2 \geq 0,$$

$$2(\langle X, Z\rangle - \|Y\|^2) + \|X\|^2 + 2\|Y\|^2 + \|Z\|^2 = \|X + Z\|^2 \geq 0.$$

If M is a minimal surface and $x \in M$, then let e_1, e_2 be an orthonormal basis of TM_x. Because M is minimal the mean curvature vector of M is zero so $0 = h(e_1, e_1) + h(e_2, e_2) = X + Z$ (X, Y, Z as above). Using $Z = -X$ in (2) yields $K(P) = \overline{K}(P) - \|X\|^2 - \|Y\|^2$ and in (1) it yields $\|h\|^2 = 2\|X\|^2 + 2\|Y\|^2$. These two equations imply Proposition 2.

If M is a totally umbilic surface, then by definition $Y = h(e_1, e_2) = 0$ and $X = h(e_1, e_1) = h(e_2, e_2) = Z$. Thus $K(P) = \overline{K}(P) + \|X\|^2$ and $\|h\|^2 = 2\|X\|^2$. This proves Proposition 3.

3. Submanifolds of pinched manifolds. If M is a Riemannian manifold and $0 < \delta \leq 1$, then M is said to be δ-pinched if and only if there is a positive constant K_0 such that $\delta K_0 \leq K(P) \leq K_0$ for all plane sections P of M. It is clear that the above results can be used to relate pinching (or holomorphic pinching) of a manifold to pinching (or holomorphic pinching) of its submanifolds. For example, Proposition 1 easily implies

PROPOSITION 5. *Let $\overline{M}$ be a Riemannian manifold with $\delta \leq \overline{K}(P) \leq 1$ for all plane sections of P of $\overline{M}$ and let M be a submanifold of $\overline{M}$ so that $\|h|_P\|^2 \leq B^2$ for all plane sections P of M. Then all the sectional curvatures of M are in the interval $[\delta - \frac{1}{2}B^2, \delta + \frac{1}{2}B^2]$. Thus if $B^2 < 2\delta$, then M is δ_M-pinched with*

$$\delta_M = \frac{\delta - B^2/2}{1 + B^2/2} = \frac{2\delta - B^2}{2 + B^2}.$$

COROLLARY. *If $\delta > \frac{1}{4}$ and M is complete with $\|h|_P\|^2 \leq (8\delta - 2)/5$ for all plane sections P of M, then M is δ_M-pinched for some $\delta_M > \frac{1}{4}$ and thus its universal covering manifold is homeomorphic to a sphere.*

We now give a statement and an elementary proof of the theorem of Flaherty mentioned above.

THEOREM [**F**]. *Let $\overline{M}$ be a complete, simply connected, Riemannian manifold of dimension at least three that has all its sectional curvatures in the interval $[\delta, 1]$ with $\delta > \frac{1}{4}$ (this implies $\overline{M}$ is homeomorphic to a sphere). Let M be a simply connected hypersurface of $\overline{M}$ such that the second fundamental forms of M with respect to one of the two outward unit normals have their eigenvalues in $[0, B]$, where $B < \cot(\pi/(4\sqrt{\delta}))$. Then M is a homotopy sphere.*

To prove this theorem we first note that if all of the eigenvalues of the second fundamental form of a hypersurface M are in the interval $[0, B]$ for one of the two choices of the outward normal, then for all plane sections P of M,

(A) $K(P) \geq \overline{K}(P)$,

(B) $\|h|_P\|^2 \leq 2B^2$.

(The first follows from the Gauss equation and the assumption that the eigenvalues are ≥ 0. For the second use that eigenvalues of $h|_P$ are also in the interval $[0, B]$ and so $\|h|_P\|^2 = \lambda_1^2 + \lambda_2^2 \leq 2B^2$.) The conditions (A) and (B) make sense for submanifolds of any codimension.

Proposition 1 now implies

PROPOSITION 6. *Let $\overline{M}$ be a Riemannian manifold with all its sectional curvatures in the interval $[\delta, 1]$ with $\delta > 0$. Let M be a complete submanifold of $\overline{M}$ that satisfies the conditions* (A) *and* (B). *Then the sectional curvatures of M are in the interval $[\delta, 1 + B^2]$ and thus M is δ_M-pinched with $\delta_M = \delta/(1 + B^2)$.*

COROLLARY. *If $\delta > \frac{1}{4}$ and $B^2 < 4\delta - 1$ in the last proposition, then M is δ_M-pinched for some $\delta_M > \frac{1}{4}$. Therefore the universal covering manifold of M is a sphere.*

To show that this corollary implies Flaherty's theorem, it is enough to show that $\frac{1}{4} < \delta \leq 1$ implies $\cot^2(\pi/(4\sqrt{\delta})) < 4\delta - 1$. Since $0 < \cot(\pi/(4\sqrt{\delta})) \leq 1$ for δ in the given interval, the required inequality is implied by $\cot(\pi/4\sqrt{\delta}) < 4\delta - 1$. Letting $x = 1\sqrt{\delta}$ we want $f(x) = 4x^{-2} - \cot(\pi x/4) - 1 > 0$ when $1 \leq x < 2$. It is enough to show f has no zero on $[1, 2)$. At a zero of f, we have $4x^{-2} - 1 = \cot(\pi x/4) \leq 1$. This inequality implies $x \geq \sqrt{2}$. Thus we only need to show $f(x) \neq 0$ on $[\sqrt{2}, 2)$. On this interval

$$\begin{aligned} f'(x) &= -\frac{8}{x^3} + \frac{\pi}{4}\csc^2\left(\frac{\pi}{4}x\right) \leq -\frac{8}{x^3}\bigg|_{x=2} + \frac{\pi}{4}\csc^2\left(\frac{\pi}{4}x\right)\bigg|_{x=\sqrt{2}} \\ &= -1.0 + .978262725 < 0. \end{aligned}$$

Therefore f is decreasing on $[\sqrt{2}, 2)$ and $f(2) = 0$. Consequently, $f(x) > 0$ on $[1, 2)$ as claimed.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH CAROLINA, COLUMBIA, SOUTH CAROLINA 29208

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, LOS ANGELES, CALIFORNIA 90024