INEQUALITIES RELATING SECTIONAL CURVATURES OF A SUBMANIFOLD TO THE SIZE OF ITS SECOND FUNDAMENTAL FORM AND APPLICATIONS TO PINCHING THEOREMS FOR SUBMANIFOLDS

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ABSTRACT. The Gauss curvature equation is used to prove inequalities relating the sectional curvatures of a submanifold with the corresponding sectional curvature of the ambient manifold and the size of the second fundamental form. These inequalities are then used to show that if a manifold $\mathcal{M}$ is $\delta$-pinched for some $\delta > \frac{1}{4}$, then any submanifold $\mathcal{M}'$ of $\mathcal{M}$ that has small enough second fundamental form is $\delta_{\mathcal{M}'}$-pinched for some $\delta_{\mathcal{M}'} > \frac{1}{4}$. It then follows from the sphere theorem that the universal covering manifold of $\mathcal{M}$ is a sphere. Some related results are also given.

1. Introduction. This note is motivated by questions of the following type: Let $\mathcal{M}$ be a complete Riemannian manifold and $\mathcal{M}'$ a compact immersed submanifold of $\mathcal{M}$; how then is the topology of $\mathcal{M}$ affected by placing a sufficiently small upper bound on the size of the second fundamental form of $\mathcal{M}$ in $\mathcal{M}'$? For example, when $\mathcal{M}$ is isometric to a standard sphere, Lawson and Simons [L-S] show that if the length of the second fundamental form of $\mathcal{M}$ is small enough, then $\mathcal{M}$ is a homotopy sphere. If $\mathcal{M}$ is the product of two spheres, then the second author has shown in [Wei] that the submanifolds of $\mathcal{M}$ with sufficiently small second fundamental are homeomorphic to totally geodesic submanifolds of $\mathcal{M}$.

Here we will consider the case that $\mathcal{M}$ is $\delta$-pinched for some $\delta > \frac{1}{4}$. That is, all sectional curvatures of $\mathcal{M}$ are in the closed interval $[\delta K_0, K_0]$ for some constant $K_0 > 0$. In this case the well-known sphere theorem of Berger, Klingenberg, Rauch and Toponogov implies that the universal covering manifold of $\mathcal{M}$ is homeomorphic to a sphere. If $\mathcal{M}$ and $\mathcal{M}'$ are both simply connected and $\mathcal{M}$ has codimension one, then Flaherty has given conditions (cf. §3 below) on the second fundamental form of $\mathcal{M}$ which forces $\mathcal{M}$ to be a homotopy sphere.

In this note we will extend this to higher codimensions and at the same time weaken the assumptions on the second fundamental form of $\mathcal{M}$ and drop the assumption of simple connectivity on $\mathcal{M}$.

Our method is to use the Gauss curvature equation to prove inequalities relating the sectional curvatures of a submanifold with the corresponding sectional curvatures of the ambient manifold and the size of the second fundamental form of the submanifold. These inequalities then imply that a submanifold of a pinched manifold is also pinched (with a slightly worse pinching constant) provided that its second fundamental form is small enough. The proofs of these inequalities are elementary; they only involve completing the square.

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2. The inequalities. Let \( M \) be an \( n \)-dimensional \( (n \geq 2) \) submanifold isometrically immersed in the Riemannian manifold \( \overline{M} \). At each point \( x \in M \) the tangent space to \( M \) at \( x \) will be written as \( TM_x \) and the normal space to \( M \) at \( x \) as \( T^\perp M_x \). The second fundamental form \( h_x \) of \( M \) in \( M \) at \( x \) is a symmetric bilinear form \( TM_x \times TM_x \to T^\perp M_x \). If \( e_1, \ldots, e_n \) is any orthonormal basis on \( TM_x \), then the length of \( h_x \) is defined by

\[
\|h_x\|^2 = \sum_{1 \leq i, j \leq n} \|h_x(e_i, e_j)\|^2.
\]

If \( P \) is a plane section of \( M \) at \( x \), i.e. a two-dimensional subspace of \( TM_x \), then denote by \( \overline{K}(P) \) the sectional curvature of \( \overline{M} \) at \( P \), by \( K(P) \) the sectional curvature of \( M \) at \( P \) and by \( h|_P \) the symmetric bilinear from \( P \times P \to T^\perp M_x \) obtained by restricting \( h_x \) to \( P \times P \). Let \( e_1, e_2 \) be any orthonormal basis of \( P \). Then the Gauss curvature equation can be written as

\[
K(P) = \overline{K}(P) + \langle h(e_1, e_1), h(e_2, e_2) \rangle - \|h(e_1, e_2)\|^2
\]

and the length of \( h|_P \) is

\[
\|h|_P\|^2 = \sum_{1 \leq i, j \leq 2} \|h(e_i, e_j)\|^2 = \|h(e_1, e_1)\|^2 + 2\|h(e_1, e_2)\|^2 + \|h(e_2, e_2)\|^2.
\]

Clearly \( \|h|_P\|^2 \leq \|h_x\|^2 \). Our estimates are

**Proposition 1.** If \( P \) is a plane section of \( M \), then

\[
\overline{K}(P) - \frac{1}{2}\|h\|^2 \leq K(P) - \frac{1}{2}\|h|_P\|^2 \leq K(P) \leq \overline{K}(P) + \frac{1}{2}\|h\|^2.
\]

**Proposition 2.** If \( M \) is a minimal surface in \( \overline{M} \), then

\[
\overline{K}(P) - \frac{1}{2}\|h\|^2 = K(P) \leq \overline{K}(P).
\]

**Proposition 3.** If \( M \) is a totally umbilic surface in \( \overline{M} \), then

\[
\overline{K}(P) \leq K(P) = \overline{K}(P) + \frac{1}{2}\|h\|^2.
\]

**Proposition 4.** If \( \overline{M} \) is a Kaehler manifold and \( M \) is a Kaehler submanifold of \( \overline{M} \), then for every holomorphic plane section \( P \) of \( M \)

\[
\overline{K}(P) - \frac{1}{2}\|h\|^2 \leq K(P) - \frac{1}{2}\|h|_P\|^2 = K(P) \leq \overline{K}(P).
\]

**Remarks.** Propositions 2 and 3 show that the inequalities in Proposition 1 are sharp in the case that \( M \) is two-dimensional. By considering cylinders over minimal surfaces or umbilic surfaces in Euclidean space it is possible to show that the inequalities in Proposition 1 are sharp in all dimensions. Proposition 4 is a restatement of Proposition 9.2 in Volume 2 of [K-N]. It is included here because of its relation to the other results.
PROOF. Let \( e_1, e_2 \) be an orthonormal basis of \( P \). Let \( X = h(e_1, e_1) \), \( Y = h(e_1, e_2) \) and \( Z = h(e_2, e_2) \). Because of equations (2) and (3), to prove Proposition 1 it is enough to show that

\[-(||X||^2 + 2||Y||^2 + ||Z||^2) \leq 2((X, Z) - ||Y||^2) \leq ||X||^2 + 2||Y||^2 + ||Z||^2.\]

This follows at once from the identities

\[||x||^2 + 2||Y||^2 + ||Z||^2 - 2((X, Z) - ||Y||^2) = ||X - Z||^2 + 4||Y||^2 \geq 0,\]

\[2((X, Z) - ||Y||^2) + ||X||^2 + 2||Y||^2 + ||Z||^2 = ||X + Z||^2 \geq 0.\]

If \( M \) is a minimal surface and \( x \in M \), then let \( e_1, e_2 \) be an orthonormal basis of \( TM_x \). Because \( M \) is minimal the mean curvature vector of \( M \) is zero so \( 0 = h(e_1, e_1) + h(e_2, e_2) = X + Z \) (\( X, Y, Z \) as above). Using \( Z = -X \) in (2) yields \( K(P) = K(P) - ||X||^2 - ||Y||^2 \) and in (1) it yields \( ||h||^2 = 2||X||^2 + 2||Y||^2 \). These two equations imply Proposition 2.

If \( M \) is a totally umbilic surface, then by definition \( Y = h(e_1, e_2) = 0 \) and \( X = h(e_1, e_1) = h(e_2, e_2) = Z \). Thus \( K(P) = K(P) + ||X||^2 \) and \( ||h||^2 = 2||X||^2 \). This proves Proposition 3.

3. Submanifolds of pinched manifolds. If \( M \) is a Riemannian manifold and \( 0 < \delta \leq 1 \), then \( M \) is said to be \( \delta \)-pinched if and only if there is a positive constant \( K_0 \) such that \( \delta K_0 \leq K(P) \leq K_0 \) for all plane sections \( P \) of \( M \). It is clear that the above results can be used to relate pinching (or holomorphic pinching) of a manifold to pinching (or holomorphic pinching) of its submanifolds. For example, Proposition 1 easily implies

\[\text{PROPOSITION 5. Let } M \text{ be a Riemannian manifold with } 0 < K(P) \leq 1 \text{ for all plane sections } P \text{ of } M \text{ and let } M \text{ be a submanifold of } M \text{ so that } ||h||^2 \leq B^2 \text{ for all plane sections } P \text{ of } M. \text{ Then all the sectional curvatures of } M \text{ are in the interval } [\delta - \frac{1}{2}B^2, \delta + \frac{1}{2}B^2]. \text{ Thus if } B^2 < 2\delta, \text{ then } M \text{ is } \delta_M \text{-pinched with}\]

\[\delta_M = \frac{\delta - B^2/2}{1 + B^2/2} = \frac{2\delta - B^2}{2 + B^2}.\]

\[\text{COROLLARY. If } \delta > \frac{1}{4} \text{ and } M \text{ is complete with } ||h||^2 \leq (8\delta - 2)/5 \text{ for all plane sections } P \text{ of } M, \text{ then } M \text{ is } \delta_M \text{-pinched for some } \delta_M > \frac{1}{4} \text{ and thus its universal covering manifold is homeomorphic to a sphere.}\]

We now give a statement and an elementary proof of the theorem of Flaherty mentioned above.

\[\text{THEOREM [F]. Let } M \text{ be a complete, simply connected, Riemannian manifold of dimension at least three that has all its sectional curvatures in the interval } [\delta, 1] \text{ with } \delta > \frac{1}{4} \text{ (this implies } M \text{ is homeomorphic to a sphere). Let } M \text{ be a simply connected hypersurface of } M \text{ such that the second fundamental forms of } M \text{ with respect to one of the two outward unit normals have their eigenvalues in } [0, B], \text{ where } B < \cot(\pi/(4\sqrt{\delta})). \text{ Then } M \text{ is a homotopy sphere.}\]

To prove this theorem we first note that if all of the eigenvalues of the second fundamental form of a hypersurface \( M \) are in the interval \([0, B]\) for one of the two
choices of the outward normal, then for all plane sections \( P \) of \( M \),
\[
\begin{align*}
\text{(A)} & \ K(P) \geq \overline{K}(P), \\
\text{(B)} & \ \|h|_{P}\|^2 \leq 2B^2.
\end{align*}
\]
(The first follows from the Gauss equation and the assumption that the eigenvalues are \( \geq 0 \). For the second use that eigenvalues of \( h|_{P} \) are also in the interval \([0, B]\) and so \( \|h|_{P}\|^2 = \lambda_1^2 + \lambda_2^2 \leq 2B^2 \).) The conditions (A) and (B) make sense for submanifolds of any codimension.

Proposition 1 now implies

**Proposition 6.** Let \( \overline{M} \) be a Riemannian manifold with all its sectional curvatures in the interval \([\delta, 1]\) with \( \delta > 0 \). Let \( M \) be a complete submanifold of \( \overline{M} \) that satisfies the conditions (A) and (B). Then the sectional curvatures of \( M \) are in the interval \([\delta, 1 + B^2]\) and thus \( M \) is \( \delta_M \)-pinched with \( \delta_M = \delta/(1 + B^2) \).

**Corollary.** If \( \delta > \frac{1}{4} \) and \( B^2 < 4\delta - 1 \) in the last proposition, then \( M \) is \( \delta_M \)-pinched for some \( \delta_M > \frac{1}{4} \). Therefore the universal covering manifold of \( M \) is a sphere.

To show that this corollary implies Flaherty’s theorem, it is enough to show that \( \frac{1}{4} < \delta \leq 1 \) implies \( \cot^2(\pi/(4\sqrt{\delta})) < 4\delta - 1 \). Since \( 0 < \cot(\pi/(4\sqrt{\delta})) \leq 1 \) for \( \delta \) in the given interval, the required inequality is implied by \( \cot(\pi/4\sqrt{\delta}) < 4\delta - 1 \). Letting \( x = 1/\sqrt{\delta} \) we want \( f(x) = 4x^{-2} - \cot(\pi x/4) - 1 \) \( > 0 \) when \( 1 < x < 2 \). It is enough to show \( f \) has no zero on \([1, 2]\). At a zero of \( f \), we have \( 4x^{-2} - 1 = \cot(\pi x/4) \leq 1 \). This inequality implies \( x \geq \sqrt{2} \). Thus we only need to show \( f(x) \neq 0 \) on \([\sqrt{2}, 2]\).

On this interval
\[
f'(x) = -\frac{8}{x^3} + \frac{\pi}{4} \csc^2(\frac{\pi}{4} x) \leq -\frac{8}{x^3} \bigg|^x=\sqrt{2} + \frac{\pi}{4} \csc^2(\frac{\pi}{4} x) \bigg|^x=\sqrt{2} = -10 + .978262725 < 0.
\]
Therefore \( f \) is decreasing on \([\sqrt{2}, 2]\) and \( f(2) = 0 \). Consequently, \( f(x) > 0 \) on \([1, 2]\) as claimed.

**REFERENCES**


