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THE HUREWICZ IMAGE OF RAY'S ELEMENTS IN MSp_*

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ABSTRACT. In this paper we determine an explicit formula for the symplectic Hurewicz homomorphism of Nigel Ray's sequence of torsion elements in the symplectic bordism ring.

Nigel Ray [7] defined a sequence of elements $\theta_n \in \pi_{4n-3}MSp$ of order two. Ray conjectured that the θ_{2n-1}, $n \geqslant 2$, are zero, and Fred Roush has an unpublished and unavailable proof of this conjecture. On the other hand, Ray proved that $\phi_0 = \theta_1$ and the $\phi_n = \theta_{2n}$, $n \geqslant 1$, are all nonzero. This sequence of elements has played a central role in all recent computations involving the torsion elements of π_*MSp either to describe them [3, 4, 6] or to eliminate them [8]. The underlying reason is that all torsion elements of π_*MSp can be organized into sequences called families which are constructed from the ϕ_n using Massey products. The details can be found in [4, §9]. Ray proved that $\{\theta_n | n \geqslant 1\}$ is closed under the Landweber-Novikov operations. From [4, §9] it follows that each family is closed under the Landweber-Novikov operations, at least modulo the Adams filtration. This gives rise to the following inductive procedure for computing differentials in the Adams spectral sequence for π_*MSp [3] and in the Atiyah-Hirzebruch spectral sequences for $\pi_*^S MSp(n)$ [2] and $\pi_*^S MSp$ [6]. The Landweber-Novikov operations induce degree lowering operations on these spectral sequences which commute with the differentials. Thus, if one inducts on the degree of members X of a fixed family, it often happens that $d_r(X)$ is determined from the knowledge of $S_E d_r(X)$ for Landweber-Novikov operations S_E. By the construction of families in [4, §9] it follows that if we knew how the S_E act on the ϕ_n then we would know how the S_E act on all families. In summary, current methods of computing with π_*MSp rely on ad hoc calculations of $S_E(\phi_n)$ for special cases of E.

In this paper we determine explicit formulas for $S_E(\theta_n)$ for all Landweber-Novikov operations S_E. Equivalently, we determined $h(\theta_n)$, where $h\colon \pi_*MSp \to MSp_*MSp$ is the Hurewicz homomorphism. Recall [1] that $MSp_*MSp(1)$ has a canonical π_*MSp-basis $\{1, b_0, b_1, \dots, b_n, \dots\}$, $\deg b_n = 4n$. Then the equation $h(x) = \sum_E x_E b_E$ is equivalent to saying that the Landweber-Novikov operations on x are given by $S_E(x) = x_E$ for all E. Here $E = (e_1, \dots, e_n)$, $b_E = b_1^{e_1} \cdots b_n^{e_n}$ and x, x_E are in π_*MSp.

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Recall that Kaoru Morisugi [6] proved the following recursive formula for $h(\theta_n)$:

$$\sum_{k=1}^{n-1} B_{n-k}^{k} h(\theta_k) + h(\theta_n) = \sum_{j=1}^{n} (n-j+1) b_{n-j} \theta_j, \tag{1}$$

where $B = 1 + b_1 + \cdots + b_t + \cdots$ and B_{n-k}^{k} denotes the component of B^k of degree $4n - 4k$. Interpret (1) as a sequence of linear equations with unknowns $h(\theta_k)$ and constant terms

$$\sum_{j=1}^{n} (n-j+1) b_{n-j} \theta_j.$$

The coefficient matrix C is lower triangular with ones on the diagonal and has (i, j)-entry $C_{ij} = B_{i-j}^{j}$ for $i > j$. We digress briefly to study C. Recall [1] that $MSp_* MSp$ is a Hopf algebra with coproduct Δ given by

$$\Delta(B) = \sum_{t=0}^{\infty} B^{t+1} \otimes b_t.$$

Thus for $i > j$,

$$\begin{aligned}
\Delta(C_{ij}) = \Delta(B)_{i-j}^{j} &= \left(\sum_{t=0}^{\infty} B^{t+1} \otimes b_t \right)_{i-j}^{j} \\
&= \left(\sum_{s_1, \ldots, s_j \geq 0} B^{s_1 + \cdots + s_j + j} \otimes b_{s_1} \cdots b_{s_j} \right)_{i-j} \\
&= \sum_{p=0}^{i-j} B_{i-j-p}^{p+j} \otimes B_p^{j} = \sum_{q=j}^{i} B_{i-q}^{q} \otimes B_{q-j}^{j} \quad \text{where } q = p + j \\
&= \sum_{q=j}^{i} C_{iq} \otimes C_{qj}.
\end{aligned}$$

Thus by [5, Lemma 2.2], Cramer's rule applies to the system of linear equations (1) to determine the unique solution as

$$h(\theta_n) = \sum_{j=1}^{n} \left[(n-j+1) b_{n-j} + \sum_{k=0}^{n-j-1} (k+1) b_k \chi(B)_{n-j-k}^{j+k} \right] \theta_j. \tag{2}$$

In this formula χ is the conjugation in the Hopf algebra $MSp_* MSp$. By [5, Lemma 2.3],

$$\chi(B)_{n-t}^{t} = \sum_{r \geq 0} \sum_{n > q_r > \cdots > q_1 > t} (-1)^{r+1} B_{n-q_r}^{q_r} B_{q_r - q_{r-1}}^{q_{r-1}} \cdots B_{q_2 - q_1}^{q_1} B_{q_1 - t}^{t}. \tag{3}$$

Taking $n = 2m$ in formula (2) we obtain

$$\begin{aligned}
h(\phi_m) = &\sum_{i=1}^{m} \left[b_{2m-2i} + \sum_{h=0}^{m-i-1} b_{2h} \chi(B)_{2m-2h-2i}^{2h+2i} \right] \phi_i \\
&+ \sum_{j=1}^{m} \left[\sum_{k=0}^{m-j} b_{2k} \chi(B)_{2m-2j-2k+1}^{2j+2k-1} \right] \theta_{2j-1}.
\end{aligned} \tag{4}$$

By Roush's unpublished result that $\theta_{2j-1} = 0$ for $j \geqslant 2$,

$$(4)' \qquad h(\phi_m) = \sum_{i=1}^{m} \left[b_{2m-2i} + \sum_{h=0}^{m-i-1} b_{2h}\chi(B)_{2m-2h-2i}^{2h+2i} \right] \phi_i + \sum_{k=0}^{m-1} b_{2k}\chi(B)_{2m-2k-1}^{2k+1}\phi_0.$$

On the other hand, one could take $n = 2m - 1$ in (2) to obtain

$$(5) \qquad h(\theta_{2m-1}) = \sum_{i=2}^{m} \left[b_{2m-2i} + \sum_{h=0}^{m-i-1} b_{2h}\chi(B)_{2m-2h-2i}^{2h+2i-1} \right] \theta_{2i-1}.$$

Note that we exclude the term $i = 1$ in this sum because

$$b_{2m-2} + \sum_{h=0}^{m-2} b_{2h}\chi(B)_{2m-2h-2}^{2h+1} = 0.$$

This is a consequence of the definition of χ and

$$\Delta(b_{2m-2}) = 1 \otimes b_{2m-2} + \sum_{h=0}^{m-2} B_{2m-2h-2}^{2h+1} \otimes b_{2h}.$$

Morisugi [**6**, Proposition 4.3] observed that the formula for $h(\theta_{2m-1})$ would not contain θ_1 nor θ_{2n}, $n \geqslant 1$. He denotes the coefficient of θ_i in $h(\theta_n)$ by $f_i(n)$. In this notation we can rewrite (2) as

$$(6) \qquad f_i(n) = (n - i + 1)b_{n-i} + \sum_{k=0}^{n-i-1} (k+1)b_k\chi(B)_{n-i-k}^{i+k}.$$

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