PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 95, Number 1, September 1985

# RELATIVE LUBIN-TATE GROUPS

EHUD DE SHALIT

ABSTRACT. We construct a class of formal groups that generalizes Lubin-Tate groups. We formulate the major properties of these groups and indicate their relation to local class field theory.

The aim of this note is to introduce a certain family of formal groups generalizing Lubin-Tate groups. Although the construction, basic properties and relation with local class field theory are all similar to Lubin-Tate theory, the author is unaware of previous references to these groups. We remark, however, that they are complementary in some sense to the formal groups studied by Honda in [**2**]. Since we want to keep this note short, all the proofs are omitted. The reader who is acquainted with Lubin-Tate theory as in [**4** or **5**] will be able to supply them without any difficulties.

I would like to acknowledge my debt to K. Iwasawa. His beautiful exposition of local class field theory [**3**] motivated this note.

**1.** Let $k$ be a finite extension of $\mathbf{Q}_p$, $\nu$: $k^\times \to \mathbf{Z}$ the normalized valuation (normalized in the sense that $\nu(k^\times) = \mathbf{Z}$), $\mathcal{O}$ and $\wp$ its ring of integers and maximal ideal, and $\bar{k} = \mathcal{O}/\wp$ the residue field, a finite field of characteristics $p$ and $q$ elements. $k^{\mathrm{alg}}$ denotes an algebraic closure of $k$ and $k^{\mathrm{ur}}$ the maximal unramified extension of $k$ in it. We also fix a completion of $k^{\mathrm{alg}}$, $\Omega$, and let $K$ be the closure of $k^{\mathrm{ur}}$ in it. We write $\varphi$ for the Frobenius automorphism of $k^{\mathrm{ur}}/k$, characterized by $\varphi(x) \equiv x^q \bmod \wp^{\mathrm{ur}}$, for all $x \in \mathcal{O}^{\mathrm{ur}}$. It extends by continuity to an automorphism of $K/k$, still denoted by $\varphi$. If $k'$ is another finite extension of $\mathbf{Q}_p$, the corresponding objects will be denoted by $'$, e.g. $\varphi'$, $q'$, etc.

If $A$ is any ring, $A[[X_1, \ldots, X_n]]$ will denote the power series ring in $X_i$. If $f$ and $g$ are elements of it, $f \equiv g$ mod deg $m$ means that the power series $f - g$ involves only monomials of degree at least $m$.

**2.** Fix the field $k$. For each integer $d$ let $\Sigma_d$ be the set of all $\xi \in k$, $\nu(\xi) = d$. Fix also $d > 0$ and let $k'$ be the unique unramified extension of $k$ of degree $d$. Let $\xi \in \Sigma_d$ and consider

$$\mathcal{F}_\xi = \{f \in \mathcal{O}'[[X]] | f \equiv \pi' X \bmod \deg 2,\ N_{k'/k}(\pi') = \xi \text{ and } f \equiv X^q \bmod \wp'\}.$$

THEOREM 1. *For each $f \in \mathcal{F}_\xi$ there is a unique one-dimensional commutative formal group law $F_f \in \mathcal{O}'[[X, Y]]$ satisfying $F_f^\varphi \circ f = f \circ F_f$. In others words, $f$ is a homomorphism of $F_f$ to $F_f^\varphi$.*

Received by the editors March 2, 1984.
1980 *Mathematics Subject Classification.* Primary 12B25.
*Key words and phrases.* Formal groups, class field theory.

Note that if $f \in \mathcal{F}_\xi$, $f^\varphi \in \mathcal{F}_\xi$ also, and necessarily $F_f^\varphi = F_{\varphi(f)}$. If $d = 1$, we are in the situation considered by Lubin and Tate. In general, we call $F_f$ a *relative Lubin-Tate* group (relative to the extension $k'/k$).

**3.**

THEOREM 2. *Let $f = \pi' X + \cdots$, $g = \pi'' X + \cdots$ be in $\mathcal{F}_\xi$. Let $a \in \mathcal{O}'$ be an element for which $a^{\varphi - 1} = \pi''/\pi'$. Then there exists a unique power series $[a]_{f,g} \in \mathcal{O}'[[X]]$ for which*
(i) $[a]_{f,g} \equiv aX \bmod \deg 2$,
(ii) $[a]_{f,g}^\varphi \circ f = g \circ [a]_{f,g}$.
*$[a]_{f,g}$ is therefore in $\mathrm{Hom}(F_f, F_g)$. If $h = \pi''' X + \cdots$ and $b^{\varphi-1} = \pi'''/\pi''$, $[ba]_{f,h} = [b]_{g,h} \circ [a]_{f,g}$. Moreover, the map $a \mapsto [a]_{f,g}$ is an additive injective homomorphism from $\{a \in \mathcal{O}' | a^{\varphi-1} = \pi''/\pi'\}$ to $\mathrm{Hom}(F_f, F_g)$. If $f = g$ it is a ring homomorphism $\mathcal{O} \to \mathrm{End}(F_f)$, $a \mapsto [a]_f = [a]_{f,f}$.*

COROLLARY. *If $f, g \in \mathcal{F}_\xi$, $F_f$ and $F_g$ are isomorphic.*

**4.** Pick $\xi, \xi' \in \Sigma_d$ and set $v = \xi/\xi'$. Let $u$ be a unit of $k'$ such that $N_{k'/k}(u) = v$, $\theta_1 \in K$ such that $\theta_1^\varphi/\theta_1 = u$, and $f \in \mathcal{F}_\xi$.

THEOREM 3. *There exists a unique power series $\theta(X) \in \mathcal{O}_K[[X]]$ satisfying*
(i) $\varphi^d(\theta) = \theta \circ [v]_f$,
(ii) $\theta(X) \equiv \theta_1 X \bmod \deg 2$.
*Put $f' = \theta^\varphi \circ f \circ \theta^{-1}$. Then $f' \in \mathcal{F}_{\xi'}$ and $\theta$ is an isomorphism of $F_f$ onto $F_{f'}$ over $\mathcal{O}_K$.*

**5.**

DEFINITION. For $i \geq 0$ and $f \in \mathcal{F}_\xi$, let $f^{(i)} = \varphi^{i-1}(f) \circ \cdots \circ \varphi(f) \circ f$. Then $f^{(i)} \in \mathrm{Hom}(F_f, F_f^{\varphi^i})$ and (if $\xi \in \Sigma_d$) $f^{(d)} = [\xi]_f \in \mathrm{End}(F_f)$. Note also that $\varphi^j(f^{(i)}) \circ f^{(j)} = f^{(i+j)}$.

Let $M$ be the valuation ideal of $\Omega$, and $M_f$ the commutative group whose underlying set is $M$ and the addition is given by $F_f$. With $\xi \in \Sigma_d$, $f \in \mathcal{F}_\xi$ and $\pi$ a prime element of $\mathcal{O}$, define for any $n \geq 0$

$$\begin{aligned} W_f^n &= \{\alpha \in M_f | [a]_f(\alpha) = 0 \text{ for all } a \in \wp^{n+1}\} \\ &= \{\alpha \in M_f | [\pi^{n+1}](\alpha) = 0\} \\ &= \mathrm{Ker}(f^{(n+1)} \colon M_f \to M_{\varphi^{n+1}(f)}). \end{aligned}$$

PROPOSITION 1. (i) *$W_f^n$ is a finite sub-$\mathcal{O}$-module of $M_f$ and has $q^{n+1}$ elements. $W_f^n \subseteq W_f^{n+1}$.*
(ii) *If $\alpha \in W_f^n$ but $\alpha \notin W_f^{n-1}$, $a \mapsto [a]_f(\alpha)$ gives an isomorphism $\mathcal{O}/\wp^{n+1} \cong W_f^n$.*
(iii) *$W_f = \bigcup W_f^n \cong k/\mathcal{O}$ (noncanonically) and is the set of all $\mathcal{O}$-torsion in $M_f$.*

**6. Coleman's norm operator (see [1]).** Let $R = \mathcal{O}'[[X]]$, $\xi \in \Sigma_d$, and $f \in \mathcal{F}_\xi$.

PROPOSITION 2. *There exists a unique multiplicative operator $\mathcal{N} \colon R \to R$ ($\mathcal{N} = \mathcal{N}_f$, to emphasize the dependence on $f$), such that*

$$(\mathcal{N} h) \circ f(X) = \prod_{\alpha \in W_f^0} h(X [+]_f \alpha) \qquad \forall h \in R.$$

*It enjoys the additional properties:*

(i) $\mathcal{N}h \equiv h^{\varphi} \bmod \wp'$,

(ii) $\mathcal{N}_f\varphi = \varphi \circ \mathcal{N}_f \circ \varphi^{-1}$, *i.e.* $\mathcal{N}_f\varphi(h^{\varphi}) = (\mathcal{N}_f h)^{\varphi}$,

(iii) *Let* $\mathcal{N}_f^{(i)}h = \mathcal{N}_{\varphi^{i-1}(f)} \circ \cdots \circ \mathcal{N}_{\varphi(f)} \circ \mathcal{N}_f(h)$.

*Then*

$$(\mathcal{N}_f^{(i)}h) \circ f^{(i)}(X) = \prod_{\alpha \in W_f^{i-1}} h(X[+]_f\alpha).$$

(iv) *If* $h \in R$ *and* $h \equiv 1 \bmod \wp'^i$ $(i \geq 1)$, *then* $\mathcal{N}h \equiv 1 \bmod \wp'^{i+1}$.

**7.**

PROPOSITION 3. *The field* $k'(W_f^n)$ *is the same for all* $f \in \mathcal{F}_\xi$. *Call it* $k_\xi^n$, *and put* $k_\xi^{-1} = k'$. *Then for* $n \geq 0$, $k_\xi^n$ *is a totally ramified extension of* $k'$ *of degree* $(q-1)q^n$, *and it is abelian over* $k$. *Any* $\alpha$ *in* $W_f^n$ *but not in* $W_f^{n-1}$, *for any* $f \in \mathcal{F}_\xi$, *generates* $k_\xi^n$ *over* $k'$ *and is a prime element for it.*

Much more can be said about those fields (see §10).

**8. Coleman power series [1].**

THEOREM 4. *Fix* $\xi \in \Sigma_d$, $f \in \mathcal{F}_\xi$ *and* $\alpha \in W^n_{\varphi^{-n}(f)}$, $\alpha \notin W^{n-1}_{\varphi^{-n}(f)}$. *For* $0 \leq i \leq n$ *let* $\alpha_i = (\varphi^{-n}(f))^{(n-i)}(\alpha) = \varphi^{-i-1}(f) \circ \cdots \circ \varphi^{-n}(f)(\alpha) \in W^i_{\varphi^{-i}(f)}$. *Let* $c$ *be a unit of* $k_\xi^n$ *and* $c_i = N_{n,i}(c)$ ($N_{n,i}$ *denoting the norm from* $k_\xi^n$ *to* $k_\xi^i$). *Then there is a power series* $g$ *in* $R$ *such that*

$$\varphi^{-i}(g)(\alpha_i) = c_i \qquad (0 \leq i \leq n).$$

COROLLARY. *Suppose* $\alpha_i$ *is an element of* $W^i_{\varphi^{-i}(f)}$ *not in* $W^{i-1}_{\varphi^{-i}(f)}$ $(i \geq 0)$ *and* $f^{\varphi^{-i}}(\alpha_i) = \alpha_{i-1}$. *Suppose also* $c_0, c_1, \ldots$ *is a norm-compatible sequence of units in* $k_\xi^i$, *i.e.* $N_{n,i}(c_n) = c_i$. *Then there exists a unique* $g$ *in* $R$ *such that* $g^{\varphi^{-i}}(\alpha_i) = c_i$ *for all* $i$.

**9.**

EXAMPLE. Let $K$ be a quadratic imaginary field, let $F$ be a finite extension of $K$, and let $E$ be an elliptic curve defined over $F$ with complex multiplication by the full ring of integers of $K$. As explained in [**6**], if we choose a Weierstrass model of $E$ over the integers of $F$ we get a formal group law $\hat{E}(X,Y)$ defined over the ring generated (over $\mathbf{Z}$) by the coefficients in the Weierstrass equation. Let $p$ be a prime of $K$ and $P$ a prime of $F$ dividing $p$. Assume $E$ has good reduction at $P$, and that $P$ is not ramified in $F/K$. It is then a consequence of the theory of complex multiplication that $\hat{E}$, as a formal group defined over $\mathcal{O}_P$ (the integers of $F_P$), is a relative Lubin-Tate group with respect to the (unramified) extension $F_P/K_p$.

**10.** The relation between Lubin-Tate groups and local class field theory can now be easily generalized. A full description of it (and actually derivation of local class field theory from the formal group point of view) can be found in [**3**]. We only make the following remarks. The fields $k_\xi = \bigcup k_\xi^n = k'(W_f)$ (for any $f \in \mathcal{F}_\xi$) are the maximal abelian extensions of $k$ with residue field equal to the extension of degree $d$ of $\bar{k}$. They are distinct for different $\xi$ as can be seen from the observation that the group of universal norms from $k_\xi$ to $k$ is just the cyclic group generated by $\xi$.

If $\xi \in \Sigma_1^d$, i.e. is a $d$th power in $k$, then $\mathcal{F}_\xi$ contains an $f$ from $\mathcal{O}[[X]]$. In this case $k_\xi$ is the compositum of a totally ramified extension of $k$ and $k'$. However, this is not always the case, because $\Sigma_d \neq \Sigma_1^d$ in general.

## REFERENCES

1. R. Coleman, *Division values in local fields*, Invent. Math. **53** (1979), 91–116.
2. T. Honda, *Formal groups and zeta functions*, Osaka. J. Math. **5** (1968) 199–213.
3. K. Iwasawa, *Local class field theory*, Oxford Univ. Press, London (to appear).
4. J. Lubin and J. Tate, *Formal complex multiplication in local fields*, Ann. of Math. (2) **81** (1965), 380–387.
5. J. P. Serre, *Local class field theory*, Algebraic Number Theory (Cassels and Frohlich, eds.), Academic Press, New York, 1967.
6. J. Tate, *The arithmetic of elliptic curves*, Invent. Math **23** (1974), 179–206.

DEPARTMENT OF MATHEMATICS, FINE HALL, PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY 08540

*Current address*: Department of Mathematics, Science Center, 1 Oxford Street, Harvard University, Cambridge, Massachusetts 02138