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RELATIVE LUBIN-TATE GROUPS

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ABSTRACT. We construct a class of formal groups that generalizes Lubin-Tate groups. We formulate the major properties of these groups and indicate their relation to local class field theory.

The aim of this note is to introduce a certain family of formal groups generalizing Lubin-Tate groups. Although the construction, basic properties and relation with local class field theory are all similar to Lubin-Tate theory, the author is unaware of previous references to these groups. We remark, however, that they are complementary in some sense to the formal groups studied by Honda in [**2**]. Since we want to keep this note short, all the proofs are omitted. The reader who is acquainted with Lubin-Tate theory as in [**4** or **5**] will be able to supply them without any difficulties.

I would like to acknowledge my debt to K. Iwasawa. His beautiful exposition of local class field theory [**3**] motivated this note.

1. Let k be a finite extension of $\mathbf{Q}_p$, ν: $k^\times \to \mathbf{Z}$ the normalized valuation (normalized in the sense that $\nu(k^\times) = \mathbf{Z}$), $\mathcal{O}$ and $\wp$ its ring of integers and maximal ideal, and $\bar{k} = \mathcal{O}/\wp$ the residue field, a finite field of characteristics p and q elements. k^{alg} denotes an algebraic closure of k and k^{ur} the maximal unramified extension of k in it. We also fix a completion of k^{alg}, Ω, and let K be the closure of k^{ur} in it. We write φ for the Frobenius automorphism of k^{ur}/k, characterized by $\varphi(x) \equiv x^q \bmod \wp^{\mathrm{ur}}$, for all $x \in \mathcal{O}^{\mathrm{ur}}$. It extends by continuity to an automorphism of K/k, still denoted by φ. If k' is another finite extension of $\mathbf{Q}_p$, the corresponding objects will be denoted by $'$, e.g. φ', q', etc.

If A is any ring, $A[[X_1, \ldots, X_n]]$ will denote the power series ring in X_i. If f and g are elements of it, $f \equiv g$ mod deg m means that the power series $f - g$ involves only monomials of degree at least m.

2. Fix the field k. For each integer d let Σ_d be the set of all $\xi \in k$, $\nu(\xi) = d$. Fix also $d > 0$ and let k' be the unique unramified extension of k of degree d. Let $\xi \in \Sigma_d$ and consider

$$\mathcal{F}_\xi = \{f \in \mathcal{O}'[[X]] | f \equiv \pi' X \bmod \deg 2,\ N_{k'/k}(\pi') = \xi \text{ and } f \equiv X^q \bmod \wp'\}.$$

THEOREM 1. *For each $f \in \mathcal{F}_\xi$ there is a unique one-dimensional commutative formal group law $F_f \in \mathcal{O}'[[X,Y]]$ satisfying $F_f^\varphi \circ f = f \circ F_f$. In others words, f is a homomorphism of F_f to F_f^φ.*

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Note that if $f \in \mathcal{F}_\xi$, $f^\varphi \in \mathcal{F}_\xi$ also, and necessarily $F_f^\varphi = F_{\varphi(f)}$. If $d = 1$, we are in the situation considered by Lubin and Tate. In general, we call F_f a *relative Lubin-Tate* group (relative to the extension k'/k).

3.

THEOREM 2. *Let $f = \pi' X + \cdots$, $g = \pi'' X + \cdots$ be in $\mathcal{F}_\xi$. Let $a \in \mathcal{O}'$ be an element for which $a^{\varphi - 1} = \pi''/\pi'$. Then there exists a unique power series $[a]_{f,g} \in \mathcal{O}'[[X]]$ for which*
(i) $[a]_{f,g} \equiv aX \bmod \deg 2$,
(ii) $[a]_{f,g}^\varphi \circ f = g \circ [a]_{f,g}$.
$[a]_{f,g}$ is therefore in $\mathrm{Hom}(F_f, F_g)$. If $h = \pi''' X + \cdots$ and $b^{\varphi-1} = \pi'''/\pi''$, $[ba]_{f,h} = [b]_{g,h} \circ [a]_{f,g}$. Moreover, the map $a \mapsto [a]_{f,g}$ is an additive injective homomorphism from $\{a \in \mathcal{O}' | a^{\varphi-1} = \pi''/\pi'\}$ to $\mathrm{Hom}(F_f, F_g)$. If $f = g$ it is a ring homomorphism $\mathcal{O} \to \mathrm{End}(F_f)$, $a \mapsto [a]_f = [a]_{f,f}$.

COROLLARY. *If $f, g \in \mathcal{F}_\xi$, F_f and F_g are isomorphic.*

4. Pick $\xi, \xi' \in \Sigma_d$ and set $v = \xi/\xi'$. Let u be a unit of k' such that $N_{k'/k}(u) = v$, $\theta_1 \in K$ such that $\theta_1^\varphi/\theta_1 = u$, and $f \in \mathcal{F}_\xi$.

THEOREM 3. *There exists a unique power series $\theta(X) \in \mathcal{O}_K[[X]]$ satisfying*
(i) $\varphi^d(\theta) = \theta \circ [v]_f$,
(ii) $\theta(X) \equiv \theta_1 X \bmod \deg 2$.
Put $f' = \theta^\varphi \circ f \circ \theta^{-1}$. Then $f' \in \mathcal{F}_{\xi'}$ and θ is an isomorphism of F_f onto $F_{f'}$ over $\mathcal{O}_K$.

5.
DEFINITION. For $i \geq 0$ and $f \in \mathcal{F}_\xi$, let $f^{(i)} = \varphi^{i-1}(f) \circ \cdots \circ \varphi(f) \circ f$. Then $f^{(i)} \in \mathrm{Hom}(F_f, F_f^{\varphi^i})$ and (if $\xi \in \Sigma_d$) $f^{(d)} = [\xi]_f \in \mathrm{End}(F_f)$. Note also that $\varphi^j(f^{(i)}) \circ f^{(j)} = f^{(i+j)}$.

Let M be the valuation ideal of Ω, and M_f the commutative group whose underlying set is M and the addition is given by F_f. With $\xi \in \Sigma_d$, $f \in \mathcal{F}_\xi$ and π a prime element of $\mathcal{O}$, define for any $n \geq 0$

$$\begin{aligned} W_f^n &= \{\alpha \in M_f | [a]_f(\alpha) = 0 \text{ for all } a \in \wp^{n+1}\} \\ &= \{\alpha \in M_f | [\pi^{n+1}](\alpha) = 0\} \\ &= \mathrm{Ker}(f^{(n+1)} \colon M_f \to M_{\varphi^{n+1}(f)}). \end{aligned}$$

PROPOSITION 1. (i) *W_f^n is a finite sub-$\mathcal{O}$-module of M_f and has q^{n+1} elements. $W_f^n \subseteq W_f^{n+1}$.*
(ii) *If $\alpha \in W_f^n$ but $\alpha \notin W_f^{n-1}$, $a \mapsto [a]_f(\alpha)$ gives an isomorphism $\mathcal{O}/\wp^{n+1} \cong W_f^n$.*
(iii) *$W_f = \bigcup W_f^n \cong k/\mathcal{O}$ (noncanonically) and is the set of all $\mathcal{O}$-torsion in M_f.*

6. Coleman's norm operator (see [1]). Let $R = \mathcal{O}'[[X]]$, $\xi \in \Sigma_d$, and $f \in \mathcal{F}_\xi$.

PROPOSITION 2. *There exists a unique multiplicative operator $\mathcal{N} \colon R \to R$ ($\mathcal{N} = \mathcal{N}_f$, to emphasize the dependence on f), such that*

$$(\mathcal{N}h) \circ f(X) = \prod_{\alpha \in W_f^0} h(X[+]_f \alpha) \qquad \forall h \in R.$$

It enjoys the additional properties:

(i) $\mathcal{N}h \equiv h^{\varphi} \bmod \wp'$,

(ii) $\mathcal{N}_f\varphi = \varphi \circ \mathcal{N}_f \circ \varphi^{-1}$, *i.e.* $\mathcal{N}_f\varphi(h^{\varphi}) = (\mathcal{N}_f h)^{\varphi}$,

(iii) *Let* $\mathcal{N}_f^{(i)}h = \mathcal{N}_{\varphi^{i-1}(f)} \circ \cdots \circ \mathcal{N}_{\varphi(f)} \circ \mathcal{N}_f(h)$.

Then

$$(\mathcal{N}_f^{(i)}h) \circ f^{(i)}(X) = \prod_{\alpha \in W_f^{i-1}} h(X[+]_f\alpha).$$

(iv) *If* $h \in R$ *and* $h \equiv 1 \bmod \wp'^i$ $(i \geq 1)$, *then* $\mathcal{N}h \equiv 1 \bmod \wp'^{i+1}$.

7.

PROPOSITION 3. *The field* $k'(W_f^n)$ *is the same for all* $f \in \mathcal{F}_\xi$. *Call it* k_ξ^n, *and put* $k_\xi^{-1} = k'$. *Then for* $n \geq 0$, k_ξ^n *is a totally ramified extension of* k' *of degree* $(q-1)q^n$, *and it is abelian over* k. *Any* α *in* W_f^n *but not in* W_f^{n-1}, *for any* $f \in \mathcal{F}_\xi$, *generates* k_ξ^n *over* k' *and is a prime element for it.*

Much more can be said about those fields (see §10).

8. Coleman power series [1].

THEOREM 4. *Fix* $\xi \in \Sigma_d$, $f \in \mathcal{F}_\xi$ *and* $\alpha \in W^n_{\varphi^{-n}(f)}$, $\alpha \notin W^{n-1}_{\varphi^{-n}(f)}$. *For* $0 \leq i \leq n$ *let* $\alpha_i = (\varphi^{-n}(f))^{(n-i)}(\alpha) = \varphi^{-i-1}(f) \circ \cdots \circ \varphi^{-n}(f)(\alpha) \in W^i_{\varphi^{-i}(f)}$. *Let* c *be a unit of* k_ξ^n *and* $c_i = N_{n,i}(c)$ ($N_{n,i}$ *denoting the norm from* k_ξ^n *to* k_ξ^i). *Then there is a power series* g *in* R *such that*

$$\varphi^{-i}(g)(\alpha_i) = c_i \qquad (0 \leq i \leq n).$$

COROLLARY. *Suppose* α_i *is an element of* $W^i_{\varphi^{-i}(f)}$ *not in* $W^{i-1}_{\varphi^{-i}(f)}$ $(i \geq 0)$ *and* $f^{\varphi^{-i}}(\alpha_i) = \alpha_{i-1}$. *Suppose also* $c_0, c_1, \ldots$ *is a norm-compatible sequence of units in* k_ξ^i, *i.e.* $N_{n,i}(c_n) = c_i$. *Then there exists a unique* g *in* R *such that* $g^{\varphi^{-i}}(\alpha_i) = c_i$ *for all* i.

9.

EXAMPLE. Let K be a quadratic imaginary field, let F be a finite extension of K, and let E be an elliptic curve defined over F with complex multiplication by the full ring of integers of K. As explained in [**6**], if we choose a Weierstrass model of E over the integers of F we get a formal group law $\hat{E}(X, Y)$ defined over the ring generated (over $\mathbf{Z}$) by the coefficients in the Weierstrass equation. Let p be a prime of K and P a prime of F dividing p. Assume E has good reduction at P, and that P is not ramified in F/K. It is then a consequence of the theory of complex multiplication that $\hat{E}$, as a formal group defined over $\mathcal{O}_P$ (the integers of F_P), is a relative Lubin-Tate group with respect to the (unramified) extension F_P/K_p.

10. The relation between Lubin-Tate groups and local class field theory can now be easily generalized. A full description of it (and actually derivation of local class field theory from the formal group point of view) can be found in [**3**]. We only make the following remarks. The fields $k_\xi = \bigcup k_\xi^n = k'(W_f)$ (for any $f \in \mathcal{F}_\xi$) are the maximal abelian extensions of k with residue field equal to the extension of degree d of $\bar{k}$. They are distinct for different ξ as can be seen from the observation that the group of universal norms from k_ξ to k is just the cyclic group generated by ξ.

If $\xi \in \Sigma_1^d$, i.e. is a dth power in k, then $\mathcal{F}_\xi$ contains an f from $\mathcal{O}[[X]]$. In this case k_ξ is the compositum of a totally ramified extension of k and k'. However, this is not always the case, because $\Sigma_d \neq \Sigma_1^d$ in general.

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