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ON RELATIVE NORMAL COMPLEMENTS IN FINITE GROUPS. III

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ABSTRACT. Let G be a finite group, let $H \leq G$, and let π be the set of prime divisors of $|H|$. Assume that whenever two elements of H are G-conjugate then they are H-conjugate. Assume that for all $h \in H^{\#}$, $(C_G(h) : C_H(h))$ is a π'-number. We prove that H is a π-Hall subgroup and that there exists a normal complement $G_0 = G - (H^{\#})^{G,\pi}$. An example shows that the generalization to relative normal complements is not true.

This paper continues the studies made in two earlier papers [**2**, **3**]. Our object here is to prove the following theorems.

THEOREM 1. *Let G be a finite group, let $H \leq G$, and let π be the set of prime divisors of $|H|$. Assume the following conditions:*

(A) *Whenever two elements of H are G-conjugate then they are H-conjugate.*

(B_1) *For every nonidentity π-element $h \in H$, $(C_G(h) : C_H(h))$ is a π'-number.*

Then H is a π-Hall subgroup of G, and there exists a unique normal complement $G_0 = G - (H^{\#})^{G,\pi}$ of H in G.

Conversely, given a π-Hall subgroup H of G and a normal complement G_0 of H, conditions (A) *and* (B_1) *must hold.*

Here we mean that $G_0 \Delta G$, $G_0 H = G$, and $G_0 \cap H = \{1\}$. The statement that H is a *π-Hall subgroup* means that $|H|$ is divisible only by primes in π and that no prime in π divides $(G : H)$. Each element x of G has a unique decomposition, $x = x_\pi x_{\pi'} = x_{\pi'} x_\pi$, into a π-element x_π and a π'-element $x_{\pi'}$. Both x_π and $x_{\pi'}$ are powers of x. Two elements x and y of G belong to the same *π-section* of G if their π-parts x_π and y_π are G-conjugate. If S is a subset of G we let $S^{G,\pi}$ denote the union of all π-sections of G that intersect S.

Under stronger assumptions than our conditions (A) and (B_1), G. Robinson has proved a theorem asserting the existence of a relative normal complement [**4**, Theorem D]. His theorem is itself a generalization of earlier results. (See his references.)

An example will show that the generalization to relative normal complements of Theorem 1 is not true. Let G be the p-group described in [**2**, §5, Example 1]. Let H and H_0 be the subgroups described there. According to [**2**, Proposition 5.1] we have

THEOREM 2. *Let G be the p-group referred to, with $H_0 \Delta H \leq G$ as described. Then:*

(i) *Whenever two elements x_1 and x_2 of H are G-conjugate then $x_1 H_0 = x_2 H_0$.*

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(ii) *If* $x \in H - H_0$ *then* $C_G(x) = H$.

(iii) *There is no relative normal complement* G_0 *of* H *over* H_0.

By definition a relative normal complement G_0 of H over H_0 would be a normal subgroup G_0 of G such that $G_0H = G$ and $G_0 \cap H = H_0$.

PROOF OF THEOREM 1. Let P be a Sylow p-subgroup of H for some $p \in \pi$. Suppose P is not a Sylow p-subgroup of G. Then some p-subgroup Q of G contains P as a proper normal subgroup. Then $Z(Q) \cap P \neq \langle 1 \rangle$. Let k be a nonidentity element of $Z(Q) \cap P$. Then $Q \subseteq C_G(k)$ but $Q \not\subseteq H$. Thus $p | (C_G(k) : C_H(k))$, contrary to hypothesis (B_1).

Thus P is a Sylow p-subgroup of G, and H is a π-Hall subgroup of G. Therefore G satisfies the hypotheses of a Theorem of Brauer [**1**, Theorem 3]. (Also see his Remark 1, p. 81.)

The converse part of our Theorem 1 follows directly from Brauer's Theorem 3 and his Remark 1 [**1**].

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