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A FOOTNOTE TO THE MULTIPLICATIVE BASIS THEOREM

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ABSTRACT. We characterize those perfect fields k such that for each integer $n \geq 1$, but there are but finitely many isomorphism types of k-algebras of dimension n that are of finite representation type. Some remarks on the imperfect case are also presented.

A finite-dimensional algebra A over a field k is of *finite representation type* if it has only finitely many isomorphism types of indecomposable modules. A *multiplicative basis* for A is a k-basis B such that $B \cup \{0\}$ is a semigroup under the multiplication in A. Recently, Bautista, Gabriel, Roiter and Salmerón [**1**] have shown that an algebraically closed field k has the property that each k-algebra of finite representation type has a multiplicative basis. Let us point out at once that this property characterizes algebraically closed fields. For, if a field k has a finite extension field F, then F is of finite representation type, and any multiplicative basis B for F would be a finite semigroup with cancellation, hence a group. Then, F would be isomorphic to the group algebra kB, but kB can never be simple if B is nontrivial.

An important consequence of the theorem above is that when k is algebraically closed, there are but finitely many k-algebras of finite representation type of any fixed k-dimension ("finite representation type is finite"). Let us express this by saying that k has property (N). We wish here to discuss other fields with property (N).

Recall from [**3**, III-29] that a field k is of *type* (F) if it is perfect and, for each $n \geq 1$, there are only finitely many k-isomorphism types of field extensions of degree n over k. Examples are finite fields, local fields with finite residue field and the fields $F((T))$ of quotients of power series over algebraically closed fields F of characteristic zero.

THEOREM. *A perfect field k has property* (N) *if and only if it is of type (F).*

PROOF. One implication is clear. If k is of type (F), let $\overline{k}$ be an algebraic closure of k, and let G be the Galois group of $\overline{k}/k$. By [**2**], a finite-dimensional k-algebra A is of finite representation type if and only if $\overline{k} \otimes_k A$ is. Hence, it suffices to show that a $\overline{k}$-algebra $\mathcal{A}$ of the form $\overline{k} \otimes_k A$ has only finitely many k-forms, up to isomorphism. Such forms are classified by the set $H^1(G, \mathrm{Aut}_{\overline{k}\text{-alg}}(\mathcal{A}))$. This set is finite by [**3**, III-30] since $\mathrm{Aut}_{\overline{k}\text{-alg}}(\mathcal{A})$ is a linear algebraic group defined over k, and the proof is complete.

Let us remark on the case of imperfect fields. The *degree of imperfection* of a field k of characteristic p is that integer r so that $[k : k^p] = p^r$ (or infinity, if $[k : k^p]$ is infinite). Thus, k is perfect just when its degree of imperfection is zero. It is well

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known that if k has degree of imperfection two or more, then k has infinitely many nonisomorphic extensions of degree p (see [**4**, II.11.6]). Thus, an imperfect field with property (N) must have degree of imperfection one. The best known examples of such fields are the function fields in one variable over a field of characteristic p. These, however, usually have many separable extensions. The most likely candidate for an imperfect field with property (N) would seem to be the separable closure of $F((T))$, with F algebraically closed of finite characteristic. It should be noted that inseparable base changes generally destroy finite representation type, so descent methods are not likely to resolve the question.

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